Seven Means, Generalized Triangular Discrimination, and Generating Divergence Measures

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Abstract

From geometrical point of view, Eve [2] studied seven means. These means are *Harmonic, Geometric, Arithmetic, Heronian, Contra-harmonic, Root-mean square* and *Centroidal mean*. We have considered for the first time a new measure calling *generalized triangular discrimination*. Inequalities among non-negative differences arising due to seven means and particular cases of *generalized triangular discrimination* are considered. Some new generating measures and their exponential representations are also presented.

Key words: *Arithmetic mean, Geometric mean, Heronian mean, Hellingar's discrimination, triangular discrimination, Information inequalities.*

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1 Seven Geometric Means

Let $a,\, b > 0$ be two positive numbers. Eves [2] studied the geometrical interpretation of the following seven means:

1. Arithmetic mean: $A(a,b) = (a+b)/2$;
2. Geometric mean: $G(a,b) = \sqrt{ab}$;
3. Harmonic mean: $H(a,b) = 2ab/(a+b)$;
4. Heronian mean: $N(a,b) = \left(a + \sqrt{ab} + b\right)\Big/3$;
5. Contra-harmonic mean: $C(a,b) = \left(a^2+b^2\right)/(a+b)$;
6. Root-mean-square: $S(a,b) = \sqrt{\left(a^2+b^2\right)/2}$;
7. Centroidal mean: $R(a,b) = 2\left(a^2+ab+b^2\right)\Big/3\,(a+b)$.

We can easily verify the following inequality having the above seven means:

$$H \leq G \leq N \leq A \leq R \leq S \leq C. \tag{1}$$

Let us write, $M(a,b) = b\, f_M(a/b)$, where M stands for any of the above seven means, then we have

$$f_H(x) \leq f_G(x) \leq f_N(x)) \leq f_A(x) \leq f_R(x) \leq f_S(x) \leq f_C(x). \tag{2}$$

where $f_H(x) = 2x/(x+1)$, $f_G(x) = \sqrt{x}$, $f_N(x) = (x+\sqrt{x}+1)/3$, $f_A(x) = (x+1)/2$, $f_R(x) = 2\left(x^2+x+1\right)/3\,(x+1)$, $f_S(x) = \sqrt{(x^2+1)/2}$ and $f_C(x) = \left(x^2+1\right)/(x+1)$, $\forall x > 0$, $x \neq 1$. In all these cases, we have equality sign iff $x = 1$, i.e., $f_{(\cdot)}(1) = 1$.

1.1 Inequalities among Differences of Means

For simplicity, let us write

$$D_{AB} = b\, f_{AB}(a,b), \tag{3}$$

where $f_{UV}(x) = f_U(x) - f_V(x)$, with $U \geq V$. Thus, according to (3), the inequality (1) admits 21 non-negative differences. These differences satisfy some simple inequalities given by the following pyramid:

$$\begin{gathered} D_{GH}; \\ D_{NG} \leq D_{NH}; \\ D_{AN} \leq D_{AG} \leq D_{AH}; \\ D_{RA} \leq D_{RN} \leq D_{RG} \leq D_{RH}; \\ D_{SR} \leq D_{SA} \leq D_{SN} \leq D_{SG} \leq D_{SH}; \\ D_{CS} \leq D_{CR} \leq D_{CA} \leq D_{CN} \leq D_{CG} \leq D_{CH}, \end{gathered}$$

where, for example, $D_{GH} := G - H$, $D_{NG} := N - G$, etc. After simplifications, we have the following equalities among some of these measures:

1. $3D_{CR} = 2D_{AH} = 2D_{CA} = D_{CH} = 6D_{RA} = \frac{3}{2}D_{RH} := \Delta$;
2. $3D_{AN} = D_{AG} = \frac{3}{2}D_{NG} := h$;
3. $D_{CG} = 3D_{RN}$.

The measures $\Delta(a,b)$ and $h(a,b)$ are the well know *triangular* and *Hellingar's discriminations* [3] given by $\Delta(a,b) = (a-b)^2/(a-b)$ and $h(a,b) = \frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2$ respectively. Not all the measures appearing in the above pyramid are convex in the the pair $(a,b) \in \mathbb{R}^2_+$. Recently, the author [12] proved the following theorem for the convex measures.

Theorem 1.1. *The following inequalities hold:*

$$D_{SA} \leq \left\{ \begin{array}{l} \frac{3}{4}D_{SN} \\ \frac{1}{3}D_{SH} \leq \frac{1}{4}\Delta \end{array} \right\} \leq \left\{ \begin{array}{l} \frac{3}{7}D_{CN} \leq \left\{ \begin{array}{l} D_{CS} \\ \frac{1}{3}D_{CG} \leq \frac{3}{5}D_{RG} \end{array} \right. \\ \frac{1}{2}D_{SG} \leq \frac{3}{5}D_{RG} \end{array} \right\} \leq h. \tag{4}$$

The proof of the above theorem is based on the following two lemmas [8, 10].

Lemma 1.1. *Let $f : I \subset \mathrm{R}_+ \to \mathrm{R}$ be a convex and differentiable function satisfying $f(1) = 0$. Consider a function*

$$\varphi_f(a,b) = af\left(\frac{b}{a}\right), \quad a, b > 0,$$

then the function $\varphi_f(a,b)$ is convex in R_+^2. Additionally, if $f'(1) = 0$, then the following inequality hold:

$$0 \leq \varphi_f(a,b) \leq \left(\frac{b-a}{a}\right)\varphi_{f'}(a,b).$$

Lemma 1.2. *Let $f_1, f_2 : I \subset \mathrm{R}_+ \to \mathrm{R}$ be two convex functions satisfying the assumptions:*
(i) $f_1(1) = f_1'(1) = 0$, $f_2(1) = f_2'(1) = 0$;
(ii) f_1 and f_2 are twice differentiable in R_+;
(iii) there exists the real constants α, β such that $0 \leq \alpha < \beta$ and

$$\alpha \leq \frac{f_1''(x)}{f_2''(x)} \leq \beta, \quad f_2''(x) > 0,$$

for all $x > 0$ then we have the inequalities:

$$\alpha\,\varphi_{f_2}(a,b) \leq \varphi_{f_1}(a,b) \leq \beta\,\varphi_{f_2}(a,b),$$

for all $a, b \in (0,\infty)$, where the function $\phi_{(\cdot)}(a,b)$ is as defined in Lemma 1.2.

1.2 Generalized Triangular Discrimination

For all $a,\ b > 0$, let consider the following measures

$$L_t(a,b) = \frac{(a-b)^2\,(a+b)^t}{2^t\left(\sqrt{ab}\right)^{t+1}}, \quad t \in \mathrm{Z} \tag{5}$$

In particular, we have

$$\begin{aligned}
L_{-1}(a,b) &= 2\,\Delta(a,b)\\
L_0(a,b) &= K(a,b) = \frac{(a-b)^2}{\sqrt{ab}},\\
L_1(a,b) &= \frac{1}{2}\Psi(a,b) = \frac{(a-b)^2\,(a+b)}{2ab},\\
L_2(a,b) &= \frac{1}{2}F(a,b) = \frac{\left(a^2-b^2\right)^2}{4\,(ab)^{3/2}}
\end{aligned}$$

and

$$L_3(a,b) = \frac{1}{8}L(a,b) = \frac{(a-b)^2\,(a+b)^3}{8\,(ab)^2}.$$

From above, we observe that the expression (5) contains some well-known measures such as $K(a,b)$ is due to Jain and Srivastava [4], $F(a,b)$ is due to Kumar and Johnson [5], $\Psi(a,b)$ is **symmetric** χ^2-**measure** [8]. While, $L(a,b)$ is considered here for the first time. More studied on related measures can be seen in [7, 9, 11, 13].

Convexity: Let us prove now the convexity of the measure (5). We can write $L_t(a,b) = b\,f(a/b)$, $t \in \mathrm{Z}$, where

$$f_{L_t}(x) = \frac{(x-1)^2 (x+1)^t}{2^t (\sqrt{x})^{t+1}}.$$

The second order derivative of the function $f_{L_t}(x)$ is given by

$$f''_{L_t}(x) = \frac{(x+1)^{t-2}}{2^{t+2} x^2 (\sqrt{x})^{t+1}} \times A_7(x,t),$$

where

$$\begin{aligned} A_7(x,t) = {} & (t+1)(t+3)\left(x^4+1\right) + \\ & + 4x\left(x^2+1\right)(2-t)(t+1) + 2x^2(3t-5)(t-1). \end{aligned} \tag{6}$$

From (6) we observe that we are unable to find unique value of t, when the function is positive. But for at least $t \in [-1,2] - (1, \frac{5}{3})$, $x > 0$, $x \neq 1$, we have $f''_{M_t}(x) \geq 0$. Also, we have $f_{L_t}(1) = 0$. Thus according to Lemma 1.1, the measure $L_t(a,b)$ is convex for all $(a,b) \in \mathrm{R}^2_+$, $t = -1, 0, 1$ and 2. Testing individually for fix $t \in \mathrm{N}$, we can check the convexity for other measures too, for example for $t = 3$, $L_3(a,b)$ is convex.

Monotonicity: Calculating the first order derivative of the function $f_{L_t}(x)$ with respect to t, we have

$$\frac{d\left(f_{L_t}(x)\right)}{dt} = \frac{(x-1)^2 (x+1)^t}{(2\sqrt{x})^{t+1}} \ln\left(\frac{(x+1)^2}{4x}\right).$$

We can easily check that for all $x > 0,\ x \neq 1$, $d\left(f_{L_t}(x)\right)/dt > 0$. This proves that the function$f_{L_t}(x)$ is decreasing with respect to t. In view of this we have

$$\tfrac{1}{4}\Delta \leq h \leq \tfrac{1}{8}K \leq \tfrac{1}{16}\Psi \leq \tfrac{1}{16}F \leq \tfrac{1}{64}L. \tag{7}$$

Also we know that $h(a,b) \leq \frac{1}{8}K(a,b)$. Thus combining (4) and (7), we have

$$D_{SA} \leq \left\{ \begin{array}{l} \frac{3}{4}D_{SN} \\ \frac{1}{3}D_{SH} \leq \frac{1}{4}\Delta \end{array} \right\} \leq \left\{ \begin{array}{l} \frac{3}{7}D_{CN} \leq \left\{ \begin{array}{l} D_{CS} \\ \frac{1}{3}D_{CG} \leq \frac{3}{5}D_{RG} \end{array} \right. \\ \frac{1}{2}D_{SG} \leq \frac{3}{5}D_{RG} \end{array} \right\} \leq$$

$$\leq h \leq \tfrac{1}{8}K \leq \tfrac{1}{16}\Psi \leq \tfrac{1}{16}F \leq \tfrac{1}{64}L. \tag{8}$$

As a part of (8), let us consider the following inequalities:

$$2\Delta \leq \tfrac{24}{7}D_{CN} \leq \tfrac{8}{3}D_{CG} \leq \tfrac{24}{5}D_{RG} \leq 8h \leq K \leq \tfrac{1}{2}\Psi \leq \tfrac{1}{2}F \leq \tfrac{1}{8}L. \tag{9}$$

Remark 1.1. *Here we have considered only the positive numbers a, $b > 0$. All the results remains valid if we consider the probability distributions $P, Q \in \Gamma_n$, where*

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \; n \geq 2.$$

(i) In terms of probability distributions the measure $L_t(a, b)$ is written as

$$L_t(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2 (p_i + q_i)^t}{2^t \left(\sqrt{p_i q_i}\right)^{t+1}}, \; (P, Q) \in \Gamma_n \times \Gamma_n, \; t \in \mathrm{Z} \tag{10}$$

(ii) Topsoe [14] considered a different type of generalized triangular discrimination

$$\Delta_t(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^{2t}}{(p_i + q_i)^{2t-1}}, \; (P, Q) \in \Gamma_n \times \Gamma_n, \; t \in \mathrm{N}_+ \tag{11}$$

In this paper our aim is to study further inequalities by considering the possible nonnegative differences arising due to (9).

2 New Inequalities

In this section we shall bring inequalities in different stages. In the first stage the measures considered are the nonnegative differences arsing due to (9). This we have done many times until we left with one measure in the final stage.

2.1 First Stage

For simplicity, let us write the expression (9) as

$$W_1 \leq W_2 \leq W_3 \leq W_4 \leq W_5 \leq W_6 \leq W_7 \leq W_8 \leq W_9, \tag{12}$$

where for example $W_1 = 2\Delta$, $W_2 = \frac{24}{7} D_{CN}$, $W_9 = \frac{8}{3} D_{CG}$, etc. We can write

$$W_t(a, b) := b\, f\left(\frac{a}{b}\right), \quad t = 1, 2, ..., 9, \tag{13}$$

where

$$f_{W_1}(x) = 2 f_{\Delta}(x) = \frac{2(x-1)^2}{x+1},$$
$$f_{W_2}(x) = \frac{24}{7} f_{CN}(x) = \frac{8\left(\sqrt{x}-1\right)^2 \left(2x + 3\sqrt{x} + 2\right)}{7(x+1)},$$
$$f_{W_3}(x) = \frac{8}{3} f_{CG}(x) = \frac{8\left(\sqrt{x}-1\right)^2 \left(x + \sqrt{x} + 1\right)}{3(x+1)},$$

$$
\begin{aligned}
f_{W_4}(x) &= \frac{24}{5} f_{RG}(x) = \frac{8\left(\sqrt{x}-1\right)^2\left(2x+\sqrt{x}+2\right)}{5\left(x+1\right)}, \\
f_{W_5}(x) &= 8 f_h(x) = 4\left(\sqrt{x}-1\right)^2, \\
f_{W_6}(x) &= f_K(x) = \frac{\left(x-1\right)^2}{\sqrt{x}}, \\
f_{W_7}(x) &= \frac{1}{2} f_{\Psi}(x) = \frac{\left(x-1\right)^2\left(x+1\right)}{2x}, \\
f_{W_8}(x) &= \frac{1}{4} f_F(x) = \frac{\left(x^2-1\right)^2}{2x^{3/2}},
\end{aligned}
$$

and

$$
f_{W_9}(x) = \frac{1}{8} f_L(x) = \frac{\left(x-1\right)^2\left(x+1\right)^3}{8x^2}.
$$

Calculating the second order derivative of above functions we have

$$
\begin{aligned}
f''_{W_1}(x) &= \frac{16}{\left(x+1\right)^3}, \\
f''_{W_2}(x) &= \frac{2\left[\left(x+1\right)^3+48x^{3/2}\right]}{7x^{3/2}\left(x+1\right)^3}, \\
f''_{W_3}(x) &= \frac{2\left[\left(x+1\right)^3+16x^{3/2}\right]}{3x^{3/2}\left(x+1\right)^3}, \\
f''_{W_4}(x) &= \frac{2\left[3\left(x+1\right)^3+16x^{3/2}\right]}{5x^{3/2}\left(x+1\right)^3}, \\
f''_{W_5}(x) &= \frac{2}{x^{3/2}}, \\
f''_{W_6}(x) &= \frac{3x^2+2x+3}{4x^{5/2}}; \\
f''_{W_7}(x) &= \frac{x^3+1}{x^3}, \\
f''_{W_8}(x) &= \frac{14x^4+2x^2+15}{16x^{7/2}},
\end{aligned}
$$

and

$$
f''_{W_9}(x) = \frac{\left(x+1\right)\left[2\left(x^4+1\right)+\left(x^2+1\right)\left(x-1\right)^2\right]}{4x^4}.
$$

The inequalities (13) again admits 45 nonnegative differences. These differences satisfies some natural inequalities given in a **pyramid** below:

$$
\begin{gathered}
D^{1}_{W_2W_1};\\
D^{2}_{W_3W_2} \leq D^{3}_{W_3W_1};\\
D^{4}_{W_4W_3} \leq D^{5}_{W_4W_2} \leq D^{6}_{W_4W_1};\\
D^{7}_{W_5W_4} \leq D^{8}_{W_5W_3} \leq D^{9}_{W_5W_2} \leq D^{10}_{W_5W_1};\\
D^{11}_{W_6W_5} \leq D^{12}_{W_6W_4} \leq D^{13}_{W_6W_3} \leq D^{14}_{W_6W_2} \leq D^{15}_{W_6W_1};\\
D^{16}_{W_7W_6} \leq D^{17}_{W_7W_5} \leq D^{18}_{W_7W_4} \leq D^{19}_{W_7W_3} \leq D^{20}_{W_7W_2} \leq D^{21}_{W_7W_1};\\
D^{22}_{W_8W_7} \leq D^{23}_{W_8W_6} \leq D^{24}_{W_8W_5} \leq D^{25}_{W_8W_4} \leq D^{26}_{W_8W_3} \leq D^{27}_{W_8W_2} \leq D^{28}_{W_8W_1};\\
D^{29}_{W_9W_8} \leq D^{30}_{W_9W_7} \leq D^{31}_{W_9W_6} \leq D^{32}_{W_9W_5} \leq D^{33}_{W_9W_4} \leq D^{34}_{W_9W_3} \leq D^{35}_{W_9W_2} \leq D^{36}_{W_9W_1}.
\end{gathered}
$$

where $D^{1}_{W_2W_1} := W_2 - W_1$, $D^{16}_{W_7W_6} := W_7 - W_6$, etc. After simplifications, we have equalities among first four lines of the **pyramid:**

$$
\begin{aligned}
&\tfrac{7}{2}D^{1}_{W_2W_1} = \tfrac{21}{8}D^{2}_{W_3W_2} = \tfrac{3}{2}D^{3}_{W_3W_1} = \tfrac{15}{8}D^{4}_{W_4W_3} = \tfrac{35}{32}D^{5}_{W_4W_2} =\\
&\quad = \tfrac{5}{6}D^{6}_{W_4W_1} = \tfrac{5}{4}D^{7}_{W_5W_4} = \tfrac{3}{4}D^{8}_{W_5W_3} = \tfrac{7}{12}D^{9}_{W_5W_2} = \tfrac{1}{2}D^{10}_{W_5W_1} = \tfrac{\left(\sqrt{a}-\sqrt{b}\right)^4}{a+b}.
\end{aligned} \tag{14}
$$

In view of above equalities we are left only with 27 nonnegative convex measures and these are connected with each other by inequalities given in the theorem below.

Theorem 2.1. *The following sequences of inequalities hold:*

$$
\begin{aligned}
&D^{1}_{W_2W_1} \leq \tfrac{1}{14}D^{15}_{W_6W_1} \leq \tfrac{1}{13}D^{14}_{W_6W_2} \leq D^{13}_{W_6W_3} \leq D^{12}_{W_6W_4} \leq D^{11}_{W_6W_5} \leq\\
&\quad \leq D^{21}_{W_7W_1} \leq D^{20}_{W_7W_2} \leq D^{19}_{W_7W_3} \leq D^{18}_{W_7W_4} \leq D^{17}_{W_7W_5} \leq D^{16}_{W_7W_6} \leq\\
&\quad\quad \leq D^{28}_{W_8W_1} \leq D^{27}_{W_8W_2} \leq D^{26}_{W_8W_3} \leq D^{25}_{W_8W_4} \leq D^{24}_{W_8W_5} \leq D^{23}_{W_8W_6} \leq \left\{ \begin{array}{c} D^{22}_{W_8W_7} \\ D^{36}_{W_9W_1} \end{array} \right\} \leq\\
&\quad\quad\quad \leq D^{35}_{W_9W_2} \leq D^{34}_{W_9W_3} \leq D^{33}_{W_9W_4} \leq D^{32}_{W_9W_5} \leq D^{31}_{W_9W_6} \leq D^{30}_{W_9W_7} \leq D^{29}_{W_9W_8}.
\end{aligned} \tag{15}
$$

Proof. We shall prove the above theorem by parts.

1. **For $\mathbf{D^{1}_{W_2W_1} \leq \frac{1}{14}D^{15}_{W_6W_1}}$:** We shall apply two approach to prove this result.
   1st Approach: Let us consider a function

$$
g_{W_2W_1\_W_6W_1}(x) = \frac{f''_{W_2W_1}(x)}{f''_{W_6W_1}(x)} \quad = \frac{f''_{W_2}(x) - f''_{W_1}(x)}{f''_{W_6}(x) - f''_{W_1}(x)}
$$

   After simplifications, we have

$$
g_{W_2W_1\_W_6W_1}(x) = \frac{8x\left(x^2 + 2x^{3/2} + 6x + 2\sqrt{x} + 1\right)}{7\left(\begin{array}{l} 3x^4 + 6x^{7/2} + 20x^3 + 34x^{5/2} + \\ +66x^2 + 34x^{3/2} + 20x + 6\sqrt{x} + 3 \end{array}\right)}
$$

$$g'_{W_2W_1\_W_6W_1}(x) = \\ = -\frac{48\,(x-1)\,(x+1)^2\begin{pmatrix} x^3+4x^{5/2}+15x^2+ \\ +20x^{3/2}+15x+4\sqrt{x}+1 \end{pmatrix}}{7\begin{pmatrix} 3x^4+6x^{7/2}+20x^3+34x^{5/2}+ \\ +66x^2+34x^{3/2}+20x+6\sqrt{x}+3 \end{pmatrix}}\begin{cases} >0 & x<1 \\ <0 & x>1 \end{cases}$$

and

$$\beta_{W_2W_1\_W_6W_1} = \sup_{x\in(0,\infty)} g_{W_2W_1\_W_6W_1}(x) = g_{W_2W_1\_W_6W_1}(1) = \frac{1}{14}.$$

By the application Lemma 1.2 we get the required result.

2nd Approach: We shall use an alternative approach to prove the above result. We know that $\beta_{W_2W_1\_W_6W_1} = g_{W_2W_1\_W_6W_1}(1) = f''_{W_2W_1}(1)/f''_{W_6W_1}(1) = \frac{1}{14}$. In order to prove the result we need to show that $\frac{1}{14}D^{15}_{W_6W_1} - D^{1}_{W_2W_1} \geq 0$. By considering the difference $\frac{1}{14}D^{15}_{W_6W_1} - D^{1}_{W_2W_1}$, we have

$$\frac{1}{14}D^{15}_{W_6W_1} - D^{1}_{W_2W_1} = \frac{1}{14}\left(W_6+13W_1-14W_2\right) = \frac{1}{14}V_1 := b\,f_{V_1}\left(\frac{a}{b}\right),$$

where

$$f_{V_1}(x) = \frac{\left(\sqrt{x}-1\right)^6}{\sqrt{x}\,(x+1)} > 0, \quad \forall x>0,\ x\neq 1. \tag{16}$$

Since $V_1(a,b) \geq 0$, we get the required result.

Note: *In two approaches applied above, we observe that the second one is easier. Moreover, in some cases we are unable to deduce the results by applications of first approach. In proving other parts, we shall only apply the second one. Without specifying, we shall frequently use the second derivatives* $f''_{W_t}(x), t = 1, 2, ..., 9$ *written above.*

2. **For $\mathbf{D^{15}_{W_6W_1} \leq \frac{14}{13}D^{14}_{W_6W_2}}$:** Let us consider a function $g_{W_6W_1\_W_6W_2}(x) = f''_{W_6W_1}(x)/f''_{W_6W_2}(x)$. After simplifications, we have

$$g_{W_6W_1\_W_6W_2}(x) = \frac{7\begin{pmatrix} 3x^4+6x^{7/2}+20x^3+34x^{5/2}+ \\ +66x^2+34x^{3/2}+20x+6\sqrt{x}+3 \end{pmatrix}}{3\begin{pmatrix} 7x^4+14x^{7/2}+44x^3+74x^{5/2}+ \\ +138x^2+74x^{3/2}+44x+14\sqrt{x}+7 \end{pmatrix}},$$

$$\beta_{W_6W_1\_W_6W_2} = g_{W_6W_1\_W_6W_2}(1) = \frac{14}{13}$$

and

$$\frac{14}{13}D^{14}_{W_6W_2} - D^{15}_{W_6W_1} = \frac{1}{13}\left(W_6+13W_1-14W_2\right) = \frac{1}{13}V_1.$$

3. **For $\mathbf{D^{14}_{W_6W_2} \leq \frac{39}{35}D^{13}_{W_6W_3}}$:** Let us consider a function $g_{W_6W_2\_W_6W_3}(x) = f''_{W_6W_2}(x)/f''_{W_6W_3}(x)$. After simplifications, we have

$$g_{W_6W_2\_W_6W_3}(x) = \frac{9}{7}\frac{\begin{pmatrix} 7x^4+14x^{7/2}+44x^3+74x^{5/2}+ \\ +138x^2+74x^{3/2}+44x+14\sqrt{x}+7 \end{pmatrix}}{\begin{pmatrix} 9x^4+18x^{7/2}+52x^3+86x^{5/2}+ \\ +150x^2+86x^{3/2}+52x+18\sqrt{x}+9 \end{pmatrix}},$$

$$\beta_{W_6W_2\_W_6W_3} = g_{W_6W_2\_W_6W_3}(1) = \frac{39}{35}$$

and
$$\frac{39}{35}D^{13}_{W_6W_3} - D^{14}_{W_6W_2} = \frac{1}{35}\left(4W_6 + 35W_2 - 39W_2\right) = \frac{4}{35}V_1.$$

4. **For $D^{13}_{W_6W_3} \leq \frac{25}{21}D^{12}_{W_6W_4}$:** Let us consider a function $g_{W_6W_3\_W_6W_4}(x) = f''_{W_6W_3}(x)/f''_{W_6W_4}(x)$. After simplifications, we have

$$g_{W_6W_3\_W_6W_4}(x) = \frac{5\left(\begin{array}{c} 9x^4 + 18x^{7/2} + 52x^3 + 86x^{5/2} + \\ +150x^2 + 86x^{3/2} + 52x + 18\sqrt{x} + 9 \end{array}\right)}{3\left(\begin{array}{c} 15x^4 + 30x^{7/2} + 76x^3 + 122x^{5/2} + \\ +186x^2 + 122x^{3/2} + 76x + 30\sqrt{x} + 15 \end{array}\right)},$$

$$\beta_{W_6W_3\_W_6W_4} = g_{W_6W_3\_W_6W_4}(1) = \frac{25}{21}$$

and
$$\frac{25}{21}D^{12}_{W_6W_4} - D^{13}_{W_6W_3} = \frac{1}{21}\left(4W_6 + 21W_3 - 25W_4\right) = \frac{4}{21}V_1.$$

5. **For $\mathbf{D^{12}_{W_6W_4} \leq \frac{7}{5}D^{11}_{W_6W_5}}$:** Let us consider a function $g_{W_6W_4\_W_6W_5}(x) = f''_{W_6W_4}(x)/f''_{W_6W_5}(x)$. After simplifications, we have

$$g_{W_6W_4\_W_6W_5}(x) = \frac{\left(\begin{array}{c} 15x^4 + 30x^{7/2} + 76x^3 + 122x^{5/2} + \\ +186x^2 + 122x^{3/2} + 76x + 30\sqrt{x} + 15 \end{array}\right)}{15\left(\sqrt{x}+1\right)^2\left(x+1\right)^3},$$

$$\beta_{W_6W_4\_W_6W_5} = g_{W_6W_4\_W_6W_5}(1) = \frac{7}{5}.$$

and
$$\frac{5}{3}D^{5}_{W_4W_2} - D^{9}_{W_5W_2} = \frac{1}{5}\left(2W_6 + 5W_4 - 7W_5\right) = \frac{2}{5}V_1 \quad .$$

6. **For $\mathbf{D^{11}_{W_6W_5} \leq \frac{1}{4}D^{21}_{W_7W_1}}$:** Let us consider a function $g_{W_6W_5\_W_7W_1}(x) = f''_{W_6W_5}(x)/f''_{W_7W_1}(x)$. After simplifications, we have

$$g_{W_6W_5\_W_7W}(x) = \frac{3\sqrt{x}\left(x+1\right)^3}{4\left(x^4 + 5x^3 + 12x^2 + 5x + 1\right)},$$

$$\beta_{W_6W_5\_W_7W_1} = g_{W_6W_5\_W_7W_1}(1) = \frac{1}{4}.$$

and
$$\frac{1}{4}D^{21}_{W_7W_1} - D^{11}_{W_6W_5} = \frac{1}{4}\left(W_7 + 4W_5 - W_1 - 4W_6\right) = \frac{1}{8}V_2 := \frac{1}{8}b f_{V_2}\left(\frac{a}{b}\right),$$

where
$$f_{V_2}(x) == \frac{\left(\sqrt{x}-1\right)^8}{x\left(x+1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{17}$$

7. **For $\mathbf{D^{21}_{W_7W_1} \leq \frac{28}{27} D^{20}_{W_7W_2}}$:** Let us consider a function $g_{W_7W_1\_W_7W_2}(x) = f''_{W_7W_1}(x)/f''_{W_7W_2}(x)$. After simplifications, we have

$$g_{W_7W_1\_W_7W_2}(x) = \frac{7\left(\sqrt{x}+1\right)^2\left(x^4+5x^3+12x^2+5x+1\right)}{\left(\begin{array}{c} 7x^5+14x^{9/2}+42x^4+68x^{7/2}+115x^3+ \\ +156x^{5/2}+115x^2+68x^{3/2}+42x+14\sqrt{x}+7 \end{array}\right)}$$

$$\beta_{W_7W_1\_W_7W_2} = g_{W_7W_1\_W_7W_2}(1) = \frac{28}{27}.$$

and

$$\frac{28}{27}D^{20}_{W_7W_2} - D^{21}_{W_7W_1} = \frac{1}{27}\left(W_7+27W_1-28W_2\right) = \frac{1}{54}V_3 := \frac{1}{54}b\, f_{V_3}\left(\frac{a}{b}\right),$$

where

$$f_{V_3}(x) = \frac{\left(x+6\sqrt{x}+1\right)\left(\sqrt{x}-1\right)^6}{x\left(x+1\right)} > 0, \quad \forall x>0,\ x \neq 1 \tag{18}$$

8. **For $\mathbf{D^{20}_{W_7W_2} \leq \frac{81}{77} D^{19}_{W_7W_3}}$:** Let us consider a function $g_{W_7W_2\_W_7W_3}(x) = f''_{W_7W_2}(x)/f''_{W_7W_3}(x)$. After simplifications, we have

$$g_{W_7W_2\_W_7W_3}(x) = \frac{3\left(\begin{array}{c} 7x^5+14x^{9/2}+42x^4+68x^{7/2}+115x^3+ \\ +156x^{5/2}+115x^2+68x^{3/2}+42x+14\sqrt{x}+7 \end{array}\right)}{7\left(\begin{array}{c} 3x^5+6x^{9/2}+18x^4+28x^{7/2}+47x^3+ \\ +60x^{5/2}+47x^2+28x^{3/2}+18x+6\sqrt{x}+3 \end{array}\right)},$$

$$\beta_{W_7W_2\_W_7W_3} = g_{W_7W_2\_W_7W_3}(1) = \frac{81}{77}.$$

and

$$\frac{81}{77}D^{19}_{W_7W_3} - D^{20}_{W_7W_2} = \frac{1}{77}\left(4W_7+77W_2-81W_3\right) = \frac{2}{77}V_3.$$

9. **For $D^{19}_{W_7W_3} \leq \frac{55}{51} D^{18}_{W_7W_4}$:** Let us consider a function $g_{W_7W_3\_W_7W_4}(x) = f''_{W_7W_3}(x)/f''_{W_7W_4}(x)$. After simplifications, we have

$$g_{W_7W_3\_W_7W_4}(x) = \frac{5\left(\begin{array}{c} 3x^5+6x^{9/2}+18x^4+28x^{7/2}+47x^3+ \\ +60x^{5/2}+47x^2+28x^{3/2}+18x+6\sqrt{x}+3 \end{array}\right)}{3\left(\begin{array}{c} 5x^5+10x^{9/2}+30x^4+44x^{7/2}+73x^3+ \\ +84x^{5/2}+73x^2+44x^{3/2}+30x+10\sqrt{x}+5 \end{array}\right)}$$

$$\beta_{W_7W_3\_W_7W_4} = g_{W_7W_3\_W_7W_4}(1) = \frac{55}{51}$$

and

$$\frac{55}{51}D^{18}_{W_7W_4} - D^{19}_{W_7W_3} = \frac{1}{51}\left(4W_7+51W_3-55W_4\right) = \frac{2}{51}V_3 \quad .$$

10. **For $\mathbf{D^{18}_{W_7W_4} \leq \frac{17}{15} D^{17}_{W_7W_5}}$:** Let us consider a function $g_{W_7W_4\_W_7W_5}(x) = f''_{W_7W_4}(x)/f''_{W_7W_5}(x)$. After simplifications, we have

$$g_{W_7W_4\_W_7W_5}(x) = \frac{\left(\begin{array}{c} 5x^5+10x^{9/2}+30x^4+44x^{7/2}+73x^3+ \\ +84x^{5/2}+73x^2+44x^{3/2}+30x+10\sqrt{x}+5 \end{array}\right)}{5\left(x+\sqrt{x}+1\right)^2\left(x+1\right)^3},$$

$$\beta_{W_7W_4\_W_7W_5} = g_{W_7W_4\_W_7W_5}(1) = \frac{17}{15}$$

and

$$\frac{17}{15}D^{17}_{W_7W_5} - D^{18}_{W_7W_4} = \frac{1}{15}\left(2W_7 + 15W_4 - 17W_5\right) = \frac{1}{15}V_3.$$

11. **For $D^{17}_{W_7W_5} \leq \frac{3}{2}D^{16}_{W_7W_6}$:** Let us consider a function $g_{W_7W_5\_W_7W_6}(x) = f''_{W_7W_5}(x)/f''_{W_7W_6}(x)$. After simplifications, we have

$$g_{W_7W_5\_W_7W_6}(x) = \frac{4\left(x+\sqrt{x}+1\right)^2}{4x^2+5x^{3/2}+6x+5\sqrt{x}+4},$$

$$\beta_{W_7W_5\_W_7W_6} = g_{W_7W_5\_W_7W_6}(1) = \frac{3}{2}$$

and

$$\frac{3}{2}D^{16}_{W_7W_6} - D^{17}_{W_7W_5} = \frac{1}{2}\left(W_7 + 2W_5 - 3W_6\right) = \frac{1}{4}V_4 := \frac{1}{4}b\,f_{V_4}\left(\frac{a}{b}\right),$$

where

$$f_{V_4}(x) = \frac{\left(\sqrt{x}-1\right)^6}{x} > 0, \quad \forall x>0,\ x\neq 1. \tag{19}$$

12. **For $\mathbf{D^{16}_{W_7W_6} \leq \frac{1}{3}D^{28}_{W_8W_1}}$:** Let us consider a function $g_{W_7W_6\_W_8W_1}(x) = f''_{W_7W_6}(x)/f''_{W_8W_1}(x)$. After simplifications, we have

$$g_{W_7W_6\_W_8W_1}(x) = \frac{4\sqrt{x}\left(x+1\right)^3\left(4x^2+5x^{3/2}+6x+5\sqrt{x}+4\right)}{\begin{pmatrix} 15+364x^{5/2}+30\sqrt{x}+492x^3+364x^{7/2}+ \\ +150x^{9/2}+90x+90x^5+30x^{11/2}+15x^6+ \\ +257x^4+257x^2+150x^{3/2} \end{pmatrix}},$$

$$\beta_{W_7W_6\_W_8W_1} = g_{W_7W_6\_W_8W_1}(1) = \frac{1}{3}$$

and

$$\frac{1}{3}D^{28}_{W_8W_1} - D^{16}_{W_7W_6} = \frac{1}{3}\left(W_8 + 3W_6 - W_1 - 3W_7\right) = \frac{1}{12}V_5 := b\,f_{V_5}\left(\frac{a}{b}\right),$$

where

$$f_{V_5}(x) = \frac{\left(\sqrt{x}+1\right)^2\left(\sqrt{x}-1\right)^8}{x^{3/2}\left(x+1\right)} > 0, \quad \forall x>0,\ x\neq 1. \tag{20}$$

13. **For $\mathbf{D^{28}_{W_8W_1} \leq \frac{42}{41}D^{27}_{W_8W_2}}$:** Let us consider a function $g_{W_8W_1\_W_8W_2}(x) = f''_{W_8W_1}(x)/f''_{W_8W_2}(x)$. After simplifications, we have

$$g_{W_8W_1\_W_8W_2}(x) = \frac{7\begin{pmatrix} 15+364x^{5/2}+30\sqrt{x}+492x^3+364x^{7/2}+ \\ +150x^{9/2}+90x+90x^5+30x^{11/2}+ \\ +15x^6+257x^4+257x^2+150x^{3/2} \end{pmatrix}}{3\begin{pmatrix} 35+828x^{5/2}+70\sqrt{x}+1084x^3+828x^{7/2}+ \\ +350x^{9/2}+210x+210x^5+70x^{11/2}+ \\ +35x^6+589x^4+589x^2+350x^{3/2} \end{pmatrix}},$$

$$\beta_{W_8W_1\_W_8W_2} = g_{W_8W_1\_W_8W_2}(1) = \frac{42}{41}$$

and

$$\frac{42}{41}D^{27}_{W_8W_2} - D^{28}_{W_8W_1} = \frac{1}{41}\left(W_8 + 41W_1 - 42W_2\right) = \frac{1}{164}V_6 := \frac{1}{164}b\,f_{V_6}\left(\frac{a}{b}\right),$$

where

$$f_{V_6}(x) = \frac{\left(x^2 + 6x^{3/2} + 22x + 6\sqrt{x} + 1\right)\left(\sqrt{x} - 1\right)^6}{x^{3/2}\left(x+1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{21}$$

14. **For $\mathbf{D^{27}_{W_8W_2} \leq \frac{123}{119}D^{26}_{W_8W_3}}$:** Let us consider a function $g_{W_8W_2\_W_8W_3}(x) = f''_{W_8W_2}(x)/f''_{W_8W_3}(x)$. After simplifications, we have

$$g_{W_8W_2\_W_8W_3}(x) = \frac{9\left(\begin{array}{l} 35 + 828x^{5/2} + 70\sqrt{x} + 1084x^3 + 828x^{7/2} + \\ +350x^{9/2} + 210x + 210x^5 + 70x^{11/2} + \\ +35x^6 + 589x^4 + 589x^2 + 350x^{3/2} \end{array}\right)}{7\left(\begin{array}{l} 45 + 1028x^{5/2} + 90\sqrt{x} + 1284x^3 + 1028x^{7/2} + \\ +450x^{9/2} + 270x + 270x^5 + 90x^{11/2} + \\ +45x^6 + 739x^4 + 739x^2 + 450x^{3/2} \end{array}\right)}$$

and

$$\beta_{W_8W_2\_W_8W_3} = g_{W_8W_2\_W_8W_3}(1) = \frac{123}{119}.$$

$$\tfrac{123}{119}D^{26}_{W_8W_3} - D^{27}_{W_8W_2} = \tfrac{1}{119}\left(4W_8 + 119W_2 - 123K_3\right) = \tfrac{1}{119}V_6.$$

15. **For $\mathbf{D^{26}_{W_8W_3} \leq \frac{85}{81}D^{25}_{W_8W_4}}$:** Let us consider a function $g_{W_8W_3\_W_8W_4}(x) = f''_{W_8W_3}(x)/f''_{W_8W_4}(x)$. After simplifications, we have

$$g_{W_8W_3\_W_8W_4}(x) = \frac{5\left(\begin{array}{l} 45 + 1028x^{5/2} + 90\sqrt{x} + 1284x^3 + 1028x^{7/2} + \\ +450x^{9/2} + 270x + 270x^5 + 90x^{11/2} + \\ +45x^6 + 739x^4 + 739x^2 + 450x^{3/2} \end{array}\right)}{3\left(\begin{array}{l} 75 + 150\sqrt{x} + 1189x^2 + 450x^5 + 1884x^3 + \\ +75x^6 + 150x^{11/2} + 450x + 1189x^4 + \\ +750x^{3/2} + 750x^{9/2} + 1628x^{7/2} + 1628x^{5/2} \end{array}\right)},$$

$$\beta_{W_8W_3\_W_8W_4} = g_{W_8W_3\_W_8W_4}(1) = \frac{85}{81}.$$

and

$$\frac{85}{81}D^{25}_{W_8W_4} - D^{26}_{W_8W_3} = 4W_8 + 81W_3 - 85W_4 = \frac{1}{81}V_6.$$

16. **For $\mathbf{D^{25}_{W_8W_4} \leq \frac{27}{25}D^{24}_{W_8W_5}}$:** Let us consider a function $g_{W_8W_4\_W_8W_5}(x) = f''_{W_8W_4}(x)/f''_{W_8W_5}(x)$. After simplifications, we have

$$g_{W_8W_4\_W_8W_5}(x) = \frac{\left(\begin{array}{l} 75 + 150\sqrt{x} + 1189x^2 + 450x^5 + 1884x^3 + \\ +75x^6 + 150x^{11/2} + 450x + 1189x^4 + \\ +750x^{3/2} + 750x^{9/2} + 1628x^{7/2} + 1628x^{5/2} \end{array}\right)}{75\left(x+1\right)^5\left(\sqrt{x}+1\right)^2},$$

$$\beta_{W_8W_4\_W_8W_5} = g_{W_8W_4\_W_8W_5}(x) = \frac{27}{25}$$

and

$$\frac{27}{25}D^{24}_{W_8W_5} - D^{25}_{W_8W_4} = \frac{1}{50}\left(2W_8 + 25W_4 - 27K_5\right) = \frac{1}{50}V_6.$$

17. **For $\mathbf{D^{24}_{W_8W_5} \leq \frac{5}{4}D^{23}_{W_8W_6}}$:** Let us consider a function $g_{W_8W_5\_W_8W_6}(x) = f''_{W_8W_5}(x)/f''_{W_8W_6}(x)$. After simplifications, we have

$$g_{W_8W_5\_W_8W_6}(x) = \frac{5\left(x+1\right)^2}{5x^2+6x+5},$$

$$\beta_{W_8W_5\_W_8W_6} = g_{W_8W_5\_W_8W_6}(1) = \frac{5}{4}$$

and

$$\frac{5}{4}D^{23}_{W_8W_6} - D^{24}_{W_8W_5} = \frac{1}{4}\left(W_8 + 4W_5 - 5W_6\right) = \frac{1}{4}V_7 := \frac{1}{4}b\, f_{V_7}\left(\frac{a}{b}\right),$$

where

$$f_{V_7}(x) = \frac{\left(x+6\sqrt{x}+1\right)\left(\sqrt{x}-1\right)^6}{x^{3/2}} > 0, \quad \forall x > 0,\ x \neq 1. \tag{22}$$

Since $f_{W_8W_5\_W_8W_6}(x) > 0$, $\forall x > 0,\ x \neq 1$, hence proving the required result.

18. **For $\mathbf{D^{23}_{W_8W_6} \leq 2D^{22}_{W_8W_7}}$:** Let us consider a function $g_{W_8W_6\_W_8W_7}(x) = f''_{W_8W_6}(x)/f''_{W_8W_7}(x)$. After simplifications, we have

$$g_{W_8W_6\_W_8W_7}(x) = \frac{\left(15x^2+18x+15\right)\left(\sqrt{x}+1\right)^2}{15x^3+14x^{5/2}+13x^2+12x^{3/2}+13x+14\sqrt{x}+15},$$

$$\beta_{W_8W_6\_W_8W_7} = g_{W_8W_6\_W_8W_7}(1) = 2.$$

and

$$2D^{22}_{W_8W_7} - D^{23}_{W_8W_6} = W_8 + W_6 - 2W_7 = \frac{1}{4}V_8 := \frac{1}{4}b\, f_{V_8}\left(\frac{a}{b}\right),$$

where

$$f_{V_8}(x) = \frac{\left(\sqrt{x}+1\right)^2\left(\sqrt{x}-1\right)^6}{x^{3/2}} > 0, \quad \forall x > 0,\ x \neq 1. \tag{23}$$

19. **For $\mathbf{D^{23}_{W_8W_6} \leq \frac{1}{2}D^{36}_{W_9W_1}}$:** Let us consider a function $g_{W_8W_6\_W_9W_1}(x) = f''_{W_8W_6}(x)/f''_{W_9W_1}(x)$. After simplifications, we have

$$g_{W_8W_6\_W_9W_1}(x) = \frac{3\sqrt{x}\left(x+1\right)^3\left(5x^2+6x+5\right)}{4\left(x+3\right)\left(3x+1\right)\left(x^4+2x^3+6x^2+2x+1\right)},$$

$$\beta_{W_8W_6\_W_9W_1} = g_{W_8W_6\_W_9W_1}(1) = \frac{1}{2}.$$

and

$$\frac{1}{2}D^{36}_{W_9W_1} - D^{23}_{W_8W_6} = \frac{1}{2}\left(W_9 + 2W_6 - K_1 - 2W_8\right) = \frac{1}{16}V_9 := \frac{1}{16}b\, f_{V_9}\left(\frac{a}{b}\right),$$

where

$$f_{V_9}(x) = \frac{\left(\sqrt{x}+1\right)^4\left(\sqrt{x}-1\right)^8}{16x^2\left(x+1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{24}$$

20. **For $\mathbf{D^{36}_{W_9W_1} \leq \frac{56}{55} D^{35}_{W_9W_2}}$:** Let us consider a function $g_{W_9W_1\_W_9W_2}(x) = f''_{W_9W_1}(x)/f''_{W_9W_2}(x)$. After simplifications, we have

$$g_{W_9W_1\_W_9W_2}(x) = \frac{7\left(x+3\right)\left(3x+1\right)\left(x^4+2x^3+6x^2+2x+1\right)\left(\sqrt{x}+1\right)^2}{\left(\begin{array}{l} 21+42\sqrt{x}+42x^{13/2}+21x^7+399x^5+775x^3+ \\ +399x^2+775x^4+960x^{7/2}+566x^{5/2}+133x+ \\ +224x^{3/2}+224x^{11/2}+566x^{9/2}+133x^6 \end{array}\right)},$$

$$\beta_{W_9W_1\_W_9W_2} = g_{W_9W_1\_W_9W_2}(x) = \frac{56}{55}$$

and

$$\frac{56}{55} D^{35}_{W_9W_2} - D^{36}_{W_9W_1} = \frac{1}{55}\left(W_9 + 55W_2 - 56K_2\right) = \frac{1}{440} V_{10} = \frac{1}{440} b\, f_{V_{10}}\left(\frac{a}{b}\right),$$

where

$$f_{V_{10}}(x) = \frac{\left(\begin{array}{l} x^3+6x^{5/2}+23x^2+ \\ +68x^{3/2}+23x+6\sqrt{x}+1 \end{array}\right)\left(\sqrt{x}-1\right)^6}{x^2\left(x+1\right)} > 0, \forall x > 0,\ x \neq 1. \tag{25}$$

21. **For $\mathbf{D^{35}_{W_9W_2} \leq \frac{165}{161} D^{34}_{W_9W_3}}$:** Let us consider a function $g_{W_9W_2\_W_9W_3}(x) = f''_{W_9W_2}(x)/f''_{W_9W_3}(x)$. After simplifications, we have

$$g_{W_9W_2\_W_9W_3}(x) = \frac{3\left(\begin{array}{l} 21+42\sqrt{x}+42x^{13/2}+21x^7+399x^5+775x^3+ \\ +399x^2+775x^4+960x^{7/2}+566x^{5/2}+133x+ \\ +224x^{3/2}+224x^{11/2}+566x^{9/2}+133x^6 \end{array}\right)}{7\left(\begin{array}{l} 9+18\sqrt{x}+9x^7+323x^4+171x^5+57x^6+ \\ +323x^3+171x^2+18x^{13/2}+96x^{11/2}+ \\ +238x^{9/2}+238x^{5/2}+57x+96x^{3/2}+384x^{7/2} \end{array}\right)},$$

$$\beta_{W_9W_2\_W_9W_3} = g_{W_9W_2\_W_9W_3}(1) = \frac{165}{161}.$$

and

$$\frac{165}{161} D^{34}_{W_9W_3} - D^{35}_{W_9W_2} = \frac{1}{161}\left(4W_9 + 161W_2 - 165W_3\right) = \frac{1}{322} V_{10}.$$

22. **For $\mathbf{D^{34}_{W_9W_3} \leq \frac{115}{111} D^{33}_{W_9W_4}}$:** Let us consider a function $g_{W_9W_3\_W_9W_4}(x) = f''_{W_9W_3}(x)/f''_{W_9W_4}(x)$. After simplifications, we have

$$g_{W_9W_3\_W_9W_4}(x) = \frac{5\left(\begin{array}{l} 9+18\sqrt{x}+9x^7+323x^4+171x^5+57x^6+ \\ +323x^3+171x^2+18x^{13/2}+96x^{11/2}+ \\ +238x^{9/2}+238x^{5/2}+57x+96x^{3/2}+384x^{7/2} \end{array}\right)}{3\left(\begin{array}{l} 15+30\sqrt{x}+95x+386x^{9/2}+576x^{7/2}+ \\ +160x^{3/2}+95x^6+30x^{13/2}+160x^{11/2}+15x^7+ \\ +517x^4+285x^5+517x^3+285x^2+386x^{5/2} \end{array}\right)},$$

$$\beta_{W_9W_3\_W_9W_4} = g_{W_9W_3\_W_9W_4}(1) = \frac{115}{111}$$

and

$$\frac{115}{111} D^{33}_{W_9W_4} - D^{34}_{W_9W_3} = \frac{1}{111}\left(4W_9 + 111W_3 - 115W_4\right) = \frac{1}{222} V_{10}.$$

23. **For $\mathbf{D^{33}_{W_9W_4} \leq \frac{37}{35} D^{32}_{W_9W_5}}$:** Let us consider a function $g_{W_9W_4\_W_9W_5}(x) = f''_{W_9W_4}(x)/f''_{W_9W_5}(x)$. After simplifications, we have

$$g_{W_9W_4\_W_9W_5}(x) = \frac{\begin{pmatrix} 15 + 30\sqrt{x} + 95x + 386x^{9/2} + 576x^{7/2} + \\ +160x^{3/2} + 95x^6 + 30x^{13/2} + 160x^{11/2} + 15x^7 + \\ +517x^4 + 285x^5 + 517x^3 + 285x^2 + 386x^{5/2} \end{pmatrix}}{5\left(x+1\right)^3 \begin{pmatrix} 3x^4 + 6x^{7/2} + 10x^3 + 14x^{5/2} + \\ +18x^2 + 14x^{3/2} + 10x + 6\sqrt{x} + 3 \end{pmatrix}},$$

$$\beta_{W_9W_4\_W_9W_5} = g_{W_9W_4\_W_9W_5}(1) = \frac{37}{35}$$

and

$$\frac{37}{35} D^{32}_{W_9W_5} - D^{33}_{W_9W_4} = \frac{1}{35}\left(2W_9 + 35W_4 - 37W_5\right) = \frac{1}{140} V_{10}.$$

24. **For $\mathbf{D^{32}_{W_9W_5} \leq \frac{7}{6} D^{31}_{W_9W_6}}$:** Let us consider a function $g_{W_9W_5\_W_9W_6}(x) = f''_{W_9W_5}(x)/f''_{W_9W_6}(x)$. After simplifications, we have

$$g_{W_9W_5\_W_9W_6}(x) = \frac{\begin{pmatrix} 3x^4 + 6x^{7/2} + 10x^3 + 14x^{5/2} + \\ +18x^2 + 14x^{3/2} + 10x + 6\sqrt{x} + 3 \end{pmatrix}}{\left(x + \sqrt{x} + 1\right) \begin{pmatrix} 3x^3 + 3x^{5/2} + 4x^2 + \\ +4x^{3/2} + 4x + 3\sqrt{x} + 3 \end{pmatrix}},$$

$$\beta_{W_9W_5\_W_9W_6} = g_{W_9W_5\_W_9W_6}(1) = \frac{7}{6}$$

and

$$\frac{7}{6} D^{31}_{W_9W_6} - D^{32}_{W_9W_5} = \frac{1}{6}\left(W_9 + 6W_5 - 7W_6\right) = \frac{1}{48} V_{11} := \frac{1}{48} b\, f_{V_{11}}\left(\frac{a}{b}\right),$$

where

$$f_{V_{11}}(x) = \frac{\left(x^2 + 6x^{3/2} + 22x + 6\sqrt{x} + 1\right)\left(\sqrt{x} - 1\right)^6}{x^2} > 0, \quad \forall x > 0,\ x \neq 1. \tag{26}$$

25. **For $\mathbf{D^{22}_{W_8W_7} \leq \frac{1}{3} D^{31}_{W_9W_6}}$:** Let us consider a function $g_{W_8W_2\_W_9W_6}(x) = f''_{W_8W_2}(x)/f''_{W_9W_6}(x)$. After simplifications, we have

$$g_{W_8W_2\_W_9W_6}(x) = \frac{\sqrt{x} \begin{pmatrix} 15x^3 + 14x^{5/2} + 13x^2 + \\ +12x^{3/2} + 13x + 14\sqrt{x} + 15 \end{pmatrix}}{4\left(x + \sqrt{x} + 1\right) \begin{pmatrix} 3x^3 + 3x^{5/2} + 4x^2 + \\ +4x^{3/2} + 4x + 3\sqrt{x} + 3 \end{pmatrix}},$$

$$\beta_{W_8W_2\_W_9W_6} = g_{W_8W_2\_W_9W_6}(1) = \frac{1}{3}$$

and

$$\frac{1}{3} D^{31}_{W_9W_6} - D^{22}_{W_8W_7} = \frac{1}{6}\left(W_9 + 3W_7 - W_6 - 3W_8\right) = \frac{1}{24} V_{12} := \frac{1}{24} b\, f_{V_3}\left(\frac{a}{b}\right),$$

where

$$f_{V_{12}}(x) = \frac{\left(\sqrt{x} + 1\right)^2 \left(\sqrt{x} - 1\right)^8}{x^2} > 0, \quad \forall x > 0,\ x \neq 1. \tag{27}$$

26. **For $\mathbf{D^{31}_{W_9W_6} \leq \frac{3}{2}D^{30}_{W_9W_7}}$:** Let us consider a function $g_{W_9W_6\_W_9W_7}(x) = f''_{W_9W_6}(x)/f''_{W_9W_7}(x)$. After simplifications, we have

$$g_{W_9W_6\_W_9W_7}(x) = \frac{(x+\sqrt{x}+1)\begin{pmatrix} 3x^3+3x^{5/2}+4x^2+ \\ +4x^{3/2}+4x+3\sqrt{x}+3 \end{pmatrix}}{3\,(x+1)\,(x^2+1)\left(\sqrt{x}+1\right)^2},$$

$$\beta_{W_9W_6\_W_9W_7} = g_{W_9W_6\_W_9W_7}(1) = \frac{3}{2}$$

and

$$\frac{3}{2}D^{30}_{W_9W_7} - D^{31}_{W_9W_6} = \frac{1}{2}\left(W_9+2W_6-3W_7\right) = \frac{1}{16}V_{13} := b\,f_{V_{13}}\left(\frac{a}{b}\right),$$

with

$$f_{V_{13}}(x) = \frac{(x-1)^2\,(x+4\sqrt{x}+1)\left(\sqrt{x}-1\right)^4}{x^2} > 0, \quad \forall x>0,\ x\neq 1. \tag{28}$$

27. **For $\mathbf{D^{30}_{W_9W_7} \leq 2D^{29}_{W_9W_8}}$:** Let us consider a function $g_{W_9W_7\_W_9W_8}(x) = f''_{W_9W_7}(x)/f''_{W_9W_8}(x)$. After simplifications, we have

$$g_{W_9W_7\_W_9W_8}(x) = \frac{12\,(x+1)\left(x^2+1\right)\left(\sqrt{x}+1\right)^2}{\begin{pmatrix} 12x^4+9x^{7/2}+10x^3+11x^{5/2}+ \\ +12x^2+11x^{3/2}+10x+9\sqrt{x}+12 \end{pmatrix}},$$

$$\beta_{W_9W_7\_W_9W_8} = g_{W_9W_7\_W_9W_8}(1) = 2$$

and

$$2D^{29}_{W_9W_8} - D^{30}_{W_9W_7} = W_9+W_7-2W_8 = \frac{1}{8}V_{14} := \frac{1}{8}b\,f_{V_{14}}\left(\frac{a}{b}\right),$$

where

$$f_{V_{14}}(x) = \frac{(x+1)\left(\sqrt{x}+1\right)^2\left(\sqrt{x}-1\right)^6}{8x^2} > 0, \quad \forall x>0,\ x\neq 1. \tag{29}$$

Combining the results 1-27, we get the proof of (15). □

Remark 2.1. *Based on the equalities given in (14) we have the following proportionality relations among the six means appearing in Section 1:*

1. $4A = 2(C+H) = 3R+H$;
2. $3R = C+2A = 2C+H$;
3. $3N = 2A+G$;
4. $3C+2H = 3R+2A$;
5. $C+6A = H+6R$;
6. $C+3N = G+3R$;
7. $3N+2A = 2C+2H+G$;
8. $27R+2G = 14A+9C+6N$;
9. $3\,(N+3R) = 8A+3C+G$;
10. $3G+8H+9C = 3R+8A+9N$;
11. $4G+14H+17C = 9R+14A+12N$;
12. $5G+24H+31C = 21R+24A+15N$.

2.2 Reverse Inequalities

We observe from the above results that the first four inequalities appearing in pyramid are equal with some multiplicative constants. The other four inequalities satisfies reverse inequalities given by

1. $D^{11}_{W_6W_5} \leq D^{12}_{W_6W_4} \leq D^{13}_{W_6W_3} \leq D^{14}_{W_6W_2} \leq D^{15}_{W_6W_1} \leq$
$\leq \frac{14}{13} D^{14}_{W_6W_2} \leq \frac{6}{5} D^{13}_{W_6W_3} \leq \frac{10}{7} D^{12}_{W_6W_4} \leq 2D^{11}_{W_6W_5};$

2. $D^{16}_{W_7W_6} \leq D^{17}_{W_7W_5} \leq D^{18}_{W_7W_4} \leq D^{19}_{W_7W_3} \leq D^{20}_{W_7W_2} \leq D^{21}_{W_7W_1} \leq$
$\leq \frac{28}{27} D^{20}_{W_7W_2} \leq \frac{12}{11} D^{19}_{W_7W_3} \leq \frac{20}{17} D^{18}_{W_7W_4} \leq \frac{4}{3} D^{17}_{W_7W_5} \leq 2D^{16}_{W_7W_6};$

3. $D^{22}_{W_8W_7} \leq D^{23}_{W_8W_6} \leq D^{24}_{W_8W_5} \leq D^{25}_{W_8W_4} \leq D^{26}_{W_8W_3} \leq D^{27}_{W_8W_2} \leq D^{28}_{W_8W_1} \leq$
$\leq \frac{42}{41} D^{27}_{W_8W_2} \leq \frac{18}{17} D^{26}_{W_8W_3} \leq \frac{10}{9} D^{25}_{W_8W_4} \leq \frac{6}{5} D^{24}_{W_8W_5} \leq \frac{3}{2} D^{23}_{W_8W_6} \leq 3D^{22}_{W_8W_7};$

4. $D^{29}_{W_9W_8} \leq D^{30}_{W_9W_7} \leq D^{31}_{W_9W_6} \leq D^{32}_{W_9W_5} \leq D^{33}_{W_9W_4} \leq D^{34}_{W_9W_3} \leq$
$\leq D^{35}_{W_9W_2} \leq D^{36}_{W_9W_1} \leq \frac{56}{55} D^{35}_{W_9W_2} \leq \frac{24}{23} D^{34}_{W_9W_3} \leq \frac{40}{37} D^{33}_{W_9W_4} \leq$
$\leq \frac{8}{7} D^{32}_{W_9W_5} \leq \frac{4}{3} D^{31}_{W_9W_6} \leq 2D^{30}_{W_9W_7} \leq 4D^{29}_{W_9W_8}.$

Remark 2.2. *It is interesting to observe that in first and second case there is difference of only 2 in between first and last elements. While, in the third case is of 3 and finally, in the forth case is of 4.*

2.3 Second Stage

In this stage we shall bring inequalities based on measures arising due to first stage. The above 27 parts generate some new measures given by

$$V_t(P||Q) := \sum_{i=1}^{n} q_i f_{V_t}\left(\frac{p_i}{q_i}\right), \quad t = 1, 2, ..., 14, \tag{30}$$

where $f_{V_t}(x)$, $t = 1, 2, ..., 14$ are as given by (16)-(29) respectively. In all the cases we have $f_{V_t}(1) = 0$, $t = 1, 2, ..., 14$. By the application of Lemma 1.1, we can say that the above 14 measures are convex. We shall try to connect 14 measures given in (30) through inequalities.

Theorem 2.2. *The following inequalities hold:*

$$V_1 \leq \tfrac{1}{8} V_3 \leq \left\{ \begin{array}{l} \frac{1}{2} V_4 \leq \frac{1}{16} V_7 \leq \frac{1}{8} V_8 \\ \frac{1}{36} V_6 \leq \frac{1}{128} V_{10} \end{array} \right\} \leq \tfrac{1}{72} V_{11} \leq \tfrac{1}{48} V_{13} \leq \tfrac{1}{16} V_{14} \tag{31}$$

and

$$V_2 \leq \tfrac{1}{4} V_5 \leq \tfrac{1}{16} V_9 \leq \tfrac{1}{8} V_{12}. \tag{32}$$

Proof. We shall prove the above theorem following the similar lines of Theorem 2.1. Since, we need the second derivatives of the functions given by (16)-(29) to prove the theorem, here below are their values:

$$f''_{V_1}(x) = \frac{\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} 3x^3 + 12x^{5/2} + 25x^2 + \\ +40x^{3/2} + 25x + 12\sqrt{x} + 3 \end{array} \right)}{4x^{5/2}\left(x+1\right)^3},$$

$$f''_{V_2}(x) = \frac{2\left(\sqrt{x}-1\right)^6 \left(\begin{array}{c} x^3 + 3x^{5/2} + 6x^2 + \\ +8x^{3/2} + 6x + 3\sqrt{x} + 1 \end{array} \right)}{x^3\left(x+1\right)^3},$$

$$f''_{V_3}(x) = \frac{2\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} x^4 + 4x^{7/2} + 13x^3 + 24x^{5/2} + \\ +36x^2 + 24x^{3/2} + 13x + 4\sqrt{x} + 1 \end{array} \right)}{x^3\left(x+1\right)^3},]$$

$$f''_{V_4}(x) = \frac{\left(\sqrt{x}-1\right)^4 \left(4x + 7\sqrt{x} + 4\right)}{4x^3},$$

$$f''_{V_5}(x) = \frac{\left(\sqrt{x}-1\right)^6 \left(\begin{array}{c} 15x^4 + 42x^{7/2} + 108x^3 + 174x^{5/2} + \\ +218x^2 + 174x^{3/2} + 108x + 42\sqrt{x} + 15 \end{array} \right)}{4x^{7/2}\left(x+1\right)^3},$$

$$f''_{V_6}(x) = \frac{5\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} 3x^5 + 12x^{9/2} + 39x^4 + 96x^{7/2} + \\ +166x^3 + 232x^{5/2} + 166x^2 + \\ +96x^{3/2} + 39x + 12\sqrt{x} + 3 \end{array} \right)}{4x^{7/2}\left(x+1\right)^3},$$

$$f''_{V_7}(x) = \frac{15\left(\sqrt{x}-1\right)^4 \left(x^2 + 4x^{3/2} + x^2 + 6x + 4\sqrt{x} + 1\right)}{4x^{7/2}},$$

$$f''_{V_8}(x) = \frac{\left(\sqrt{x}-1\right)^4 \left(15x^2 + 28x^{3/2} + 34x + 28\sqrt{x} + 15\right)}{4x^{7/2}},$$

$$f''_{V_9}(x) = \frac{\left(\sqrt{x}+1\right)^2\left(\sqrt{x}-1\right)^6 \left(\begin{array}{c} 6x^4 + 9x^{7/2} + 32x^3 + 35x^{5/2} + \\ +60x^2 + 35x^{3/2} + 32x^2 + 9\sqrt{x} + 6 \end{array} \right)}{x^4\left(x+1\right)^3},$$

$$f''_{V_{10}}(x) = \frac{2\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} 3x^6 + 12x^{11/2} + 40x^5 + 100x^{9/2} + 217x^4 + \\ +352x^{7/2} + 472x^3 + 352x^{5/2} + 217x^2 + \\ +100x^{3/2} + 40x + 12\sqrt{x} + 3 \end{array} \right)}{x^4\left(x+1\right)^3},$$

$$f''_{V_{11}}(x) = \frac{2\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} 3x^3 + 12x^{5/2} + 31x^2 + \\ +43x^{3/2} + 31x + 12\sqrt{x} + 3 \end{array} \right)}{x^4}$$

$$f''_{V_{12}}(x) = \frac{\left(\sqrt{x}-1\right)^6 \left(12x^2 + 27x^{3/2} + 34x + 27\sqrt{x} + 12\right)}{2x^4},$$

$$f''_{V_{13}}(x) = \frac{2\left(\sqrt{x}-1\right)^4 \left(\begin{array}{c} 3x^3 + 12x^{5/2} + 19x^2 + \\ +22x^{3/2} + 19x + 12\sqrt{x} + 3 \end{array} \right)}{x^4}$$

and

$$f''_{V_{14}}(x) = \frac{(\sqrt{x}-1)^4 \begin{pmatrix} 6x^3 + 9x^{5/2} + 10x^2 + \\ +10x^{3/2} + 10x + 9\sqrt{x} + 6 \end{pmatrix}}{x^4}.$$

We shall prove the above theorem by parts. In view of procedure used in Theorem 2.1, we shall write the proof of each part in very summarized way.

1. **For $\mathbf{V_1(a,b) \leq \frac{1}{8}V_3(a,b)}$:** Let us consider a function $g_{V_1\_V_3}(x) = f''_{V_1}(x) / f''_{V_3}(x)$. After simplifications, we have

$$g_{V_1\_V_3}(x) = \frac{x \begin{pmatrix} 3x^3 + 12x^{5/2} + 25x^2 + \\ +40x^{3/2} + 25x + 12\sqrt{x} + 3 \end{pmatrix}}{8 \begin{pmatrix} 24x^3 + \sqrt{x} + 36x^{5/2} + 13x^{3/2} + \\ +24x^2 + 4x + 13x^{7/2} + x^{9/2} + 4x^4 \end{pmatrix}},$$

$$\beta_{V_1\_V_3} = g_{V_1\_V_3}(1) = \frac{1}{8}$$

and

$$\tfrac{1}{8}V_3(a,b) - V_1(a,b) = \tfrac{1}{8}U_1(a,b) = \tfrac{1}{8}b\, f_{U_1}\left(\tfrac{a}{b}\right),$$

where

$$f_{U_1}(x) = \frac{(\sqrt{x}-1)^8}{x\,(x+1)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{33}$$

2. **For $\mathbf{V_3(a,b) \leq 4V_4(a,b)}$:** Let us consider a function $g_{V_3\_V_4}(x) = f''_{V_3}(x) / f''_{V_4}(x)$. After simplifications, we have

$$g_{V_3\_V_4}(x) = \frac{4 \begin{pmatrix} 24x^3 + \sqrt{x} + 36x^{5/2} + 13x^{3/2} + \\ +24x^2 + 4x + 13x^{7/2} + x^{9/2} + 4x^4 \end{pmatrix}}{\left(4x^{3/2} + 7x + 4\sqrt{x}\right)(x+1)^3},$$

$$\beta_{V_3\_V_4} = g_{V_3\_V_4}(1) = 4$$

and

$$4V_4(a,b) - V_3(a,b) = 3U_1(a,b).$$

3. **For $\mathbf{V_4(a,b) \leq \frac{1}{8}V_7(a,b)}$:** Let us consider a function $g_{V_4\_V_7}(x) = f''_{V_4}(x) / f''_{V_7}(x)$. After simplifications, we have

$$g_{V_4\_V_7}(x) = \frac{2\left(4x^{3/2} + 7x + 4\sqrt{x}\right)}{15\left(4\sqrt{x} + 6x + 1 + x^2 + 4x^{3/2}\right)},$$

$$\beta_{V_4\_V_7} = g_{V_4\_V_7}(1) = \frac{1}{8}$$

and

$$\frac{1}{8}V_7(a,b) - V_4(a,b) = \frac{1}{8}U_2(a,b) = \frac{1}{8}b\, f_{U_2}\left(\frac{a}{b}\right),$$

where

$$f_{U_2}(x) = \frac{(\sqrt{x}-1)^8}{x^{3/2}} > 0, \quad \forall x > 0,\ x \neq 1. \tag{34}$$

4. **For $\mathbf{V_3(a,b) \leq \frac{2}{9}V_6(a,b)}$:** Let us consider a function $g_{V_3\_V_6}(x) = f''_{V_3}(x)/f''_{V_6}(x)$. After simplifications, we have

$$g_{V_3\_V_6}(x) = \frac{8\begin{pmatrix} 24x^3 + \sqrt{x} + 36x^{5/2} + 13x^{3/2} + \\ +24x^2 + 4x + 13x^{7/2} + x^{9/2} + 4x^4 \end{pmatrix}}{5\begin{pmatrix} 166x^2 + 96x^{3/2} + 12\sqrt{x} + 39x + 3 + 3x^5 + \\ +232x^{5/2} + 12x^{9/2} + 96x^{7/2} + 166x^3 + 39x^4 \end{pmatrix}},$$

$$\beta_{V_3\_V_6} = g_{V_3\_V_6}(1) = \frac{2}{9}$$

and

$$\frac{2}{9}V_6(a,b) - V_3(a,b) = \frac{1}{9}U_3(a,b) = \frac{1}{9}b\, f_{U_3}\left(\frac{a}{b}\right),$$

where

$$f_{U_3}(x) = \frac{(\sqrt{x}-1)^8\,(2x + 7\sqrt{x} + 2)}{x^{3/2}\,(x+1)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{35}$$

5. **For $\mathbf{V_6(a,b) \leq \frac{9}{32}V_{10}(a,b)}$:** Let us consider a function $g_{V_6\_V_{10}}(x) = f''_{V_6}(x)/f''_{V_{10}}(x)$. After simplifications, we have

$$g_{V_6\_V_{10}}(x) = \frac{5x\begin{pmatrix} 166x^2 + 96x^{3/2} + 12\sqrt{x} + 39x + 3 + 3x^5 + \\ +232x^{5/2} + 12x^{9/2} + 96x^{7/2} + 166x^3 + 39x^4 \end{pmatrix}}{8\begin{pmatrix} 12x^6 + 100x^2 + 40x^{3/2} + 3\sqrt{x} + 12x + \\ +100x^5 + 217x^{5/2} + 217x^{9/2} + 472x^{7/2} + \\ +352x^3 + 352x^4 + 40x^{11/2} + 3x^{13/2} \end{pmatrix}},$$

This gives $\beta_{V_6\_V_{10}} = g_{V_6\_V_{10}}(1) = \frac{9}{32}$. Let us consider now,

and

$$\frac{9}{32}V_{10}(a,b) - V_6(a,b) = \frac{1}{32}U_4(a,b) = \frac{1}{32}b\, f_{U_4}\left(\frac{a}{b}\right),$$

where

$$f_{U_4}(x) = \frac{(\sqrt{x}-1)^8\left(9x^2 + 40x^{3/2} + 86x + 40\sqrt{x} + 9\right)}{x^2\,(x+1)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{36}$$

6. **For $\mathbf{V_{10}(a,b) \leq \frac{16}{9}V_{11}(a,b)}$:** Let us consider a function $g_{V_{10}\_V_{11}}(x) = f''_{V_{10}}(x)/f''_{V_{11}}(x)$. After simplifications, we have

$$g_{V_{10}\_V_{11}}(x) = \frac{\begin{pmatrix} 12x^6 + 100x^2 + 40x^{3/2} + 3\sqrt{x} + 12x + \\ +100x^5 + 217x^{5/2} + 217x^{9/2} + 472x^{7/2} + \\ +352x^3 + 352x^4 + 40x^{11/2} + 3x^{13/2} \end{pmatrix}}{\begin{pmatrix} 43x^2 + 31x^{3/2} + 3\sqrt{x} + \\ +12x + 31x^{5/2} + 3x^{7/2} + 12x^3 \end{pmatrix}(x+1)^3},$$

$$\beta_{V_{10}\_V_{11}} = g_{V_{10}\_V_{11}}(1) = \frac{16}{9}$$

and

$$\frac{16}{9}V_{11}(a,b) - V_{10}(a,b) = \frac{7}{9}U_5(a,b) = \frac{7}{9}b\,f_{U_5}\left(\frac{a}{b}\right),$$

where

$$f_{U_5}(x) = \frac{\left(\sqrt{x}-1\right)^8\left(x^2+8x^{3/2}+38x+8\sqrt{x}+1\right)}{x^2\left(x+1\right)} > 0, \quad \forall x>0,\ x\neq 1. \tag{37}$$

7. **For $\mathbf{V_6(a,b) \leq \frac{9}{4}V_7(a,b)}$:** Let us consider a function $g_{V_6\_V_7}(x) = f''_{V_6}(x)/f''_{V_7}(x)$. After simplifications, we have

$$g_{V_6\_V_7}(x) = \frac{\left(\begin{array}{c} 166x^2+96x^{3/2}+12\sqrt{x}+39x+3+3x^5+ \\ +232x^{5/2}+12x^{9/2}+96x^{7/2}+166x^3+39x^4 \end{array}\right)}{3\left(4\sqrt{x}+6x+1+x^2+4x^{3/2}\right)\left(x+1\right)^3},$$

$$\beta_{V_6\_V_7} = g_{V_6\_V_7}(1) = \frac{9}{4}$$

and

$$\frac{9}{4}V_7(a,b) - V_6(a,b) = \frac{5}{4}U_6(a,b) = \frac{5}{4}b\,f_{U_6}\left(\frac{a}{b}\right),$$

where

$$f_{U_6}(x) = \frac{\left(\sqrt{x}-1\right)^8\left(x+8\sqrt{x}+1\right)}{x^{3/2}\left(x+1\right)} > 0, \quad \forall x>0,\ x\neq 1. \tag{38}$$

8. **For $\mathbf{V_7(a,b) \leq 2\,V_8(a,b)}$:** Let us consider a function $g_{V_7\_V_8}(x) = f''_{V_7}(x)/f''_{V_8}(x)$. After simplifications, we have

$$g_{V_7\_V_8}(x) = \frac{15\left(4\sqrt{x}+6x+1+x^2+4x^{3/2}\right)}{15x^2+28x^{3/2}+34x+28\sqrt{x}+15},$$

$$\beta_{V_7\_V_8} = g_{V_7\_V_8}(1) = 2$$

and

$$2V_8(a,b) - V_7 = U_2(a,b).$$

9. **For $\mathbf{V_8(a,b) \leq \frac{1}{9}\,V_{11}(a,b)}$:** Let us consider a function $g_{V_8\_V_{11}}(x) = f''_{V_8}(x)/f''_{V_{11}}(x)$. After simplifications, we have

$$g_{V_8\_V_{11}}(x) = \frac{x\left(15x^2+28x^{3/2}+34x+28\sqrt{x}+15\right)}{8\left(\begin{array}{c} 43x^2+31x^{3/2}+3\sqrt{x}+ \\ +12x+31x^{5/2}+3x^{7/2}+12x^3 \end{array}\right)},$$

$$\beta_{V_8\_V_{11}} = g_{V_8\_V_{11}}(1) = \frac{1}{9}$$

and

$$4V_2(a,b) - V_4(a,b) = \frac{1}{9}U_7(a,b) = \frac{1}{9}b\,f_{U_7}\left(\frac{a}{b}\right),$$

where

$$f_{U_7}(x) = \frac{\left(\sqrt{x}-1\right)^8\left(x-\sqrt{x}+1\right)}{x^2} > 0, \quad \forall x>0,\ x\neq 1. \tag{39}$$

10. **For $\mathbf{V_{11}(a,b) \leq \frac{3}{2}\,V_{13}(a,b)}$:** Let us consider a function $g_{V_{11}\_V_{13}}(x) = f''_{V_{11}}(x)/f''_{V_{13}}(x)$. After simplifications, we have

$$g_{V_{11}\_V_{13}}(x) = \frac{\begin{pmatrix} 43x^2 + 31x^{3/2} + 3\sqrt{x} + 12x + \\ +31x^{5/2} + 3x^{7/2} + 12x^3 \end{pmatrix}}{\sqrt{x}\begin{pmatrix} 3x^3 + 12x^{5/2} + 19x^2 + \\ +22x^{3/2} + 19x + 12\sqrt{x} + 3 \end{pmatrix}},$$

$$\beta_{V_{11}\_V_{13}} = g_{V_{11}\_V_{13}}(1) = \frac{3}{2}$$

and

$$\frac{3}{2}V_{13}(a,b) - V_{11}(a,b) = \frac{1}{2}U_8(a,b) = \frac{1}{2}b\,f_{U_8}\left(\frac{a}{b}\right),$$

where

$$f_{U_8}(x) = \frac{\left(\sqrt{x}-1\right)^8\left(x + 8\sqrt{x} + 1\right)}{x^2} > 0, \quad \forall x > 0,\ x \neq 1. \tag{40}$$

11. **For $\mathbf{V_{13}(a,b) \leq 3\,V_{14}(a,b)}$:** Let us consider a function $g_{V_{13}\_V_{14}}(x) = f''_{V_{13}}(x)/f''_{V_{14}}(x)$. After simplifications, we have

$$g_{V_{13}\_V_{14}}(x) = \frac{2\begin{pmatrix} 3x^3 + 12x^{5/2} + 19x^2 + \\ +22x^{3/2} + 19x + 12\sqrt{x} + 3 \end{pmatrix}}{\begin{pmatrix} 10x^{3/2} + 10x + 9\sqrt{x} + 6 + \\ +10x^2 + 9x^{5/2} + 6x^3 \end{pmatrix}},$$

$$\beta_{V_{13}\_V_{14}} = g_{V_{13}\_V_{14}}(1) = 3$$

and

$$3V_{14}(a,b) - V_{13}(a,b) = 2U_9(a,b) = 2b\,f_{U_9}\left(\frac{a}{b}\right),$$

where

$$f_{U_9}(x) = \frac{\left(\sqrt{x}-1\right)^8\left(\sqrt{x}+1\right)^2}{x^2} > 0, \quad \forall x > 0,\ x \neq 1. \tag{41}$$

12. **For $\mathbf{V_2(a,b) \leq \frac{1}{4}\,V_5(a,b)}$:** Let us consider a function $g_{V_2\_V_5}(x) = f''_{V_2}(x)/f''_{V_5}(x)$. After simplifications, we have

$$g_{V_2\_V_5}(x) = \frac{8\left(3x^3 + 6x^{3/2} + 3x + 8x^2 + x^{7/2} + 6x^{5/2} + \sqrt{x}\right)}{\begin{pmatrix} 218x^2 + 174x^{3/2} + 42\sqrt{x} + 108x + \\ +15 + 174x^{5/2} + 42x^{7/2} + 108x^3 + 15x^4 \end{pmatrix}},$$

$$\beta_{V_2\_V_5} = g_{V_2\_V_5}(1) = \frac{1}{4}$$

and

$$\frac{1}{4}V_5(a,b) - V_2(a,b) = \frac{1}{4}U_{10}(a,b) = \frac{1}{4}b\,f_{U_{10}}\left(\frac{a}{b}\right),$$

where

$$f_{U_{10}}(x) = \frac{\left(\sqrt{x}-1\right)^{10}}{x^{3/2}\left(x+1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{42}$$

13. **For $\mathbf{V_5(a,b) \leq \frac{1}{4}\,V_9(a,b)}$:** Let us consider a function $g_{V_5\_V_9}(x) = f''_{V_5}(x)/f''_{V_9}(x)$. After simplifications, we have

$$g_{V_4\_V_2}(x) = \frac{x\left(\begin{array}{c} 218x^2 + 174x^{3/2} + 42\sqrt{x} + 108x + \\ +15 + 174x^{5/2} + 42x^{7/2} + 108x^3 + 15x^4 \end{array}\right)}{4\left(\sqrt{x}+1\right)^2\left(\begin{array}{c} 35x^2 + 32x^{3/2} + 6\sqrt{x} + 9x + 60x^{5/2} + \\ +6x^{9/2} + 32x^{7/2} + 35x^3 + 9x^4 \end{array}\right)},$$

$$\beta_{V_5\_V_9} = g_{V_5\_V_9}(1) = \frac{1}{4}$$

and

$$4V_2(a,b) - V_4(a,b) = \frac{1}{4}U_{11}(a,b) = \frac{1}{4}b\,f_{U_{11}}\left(\frac{a}{b}\right),$$

where

$$f_{U_{11}}(x) = \frac{\left(\sqrt{x}+1\right)^2\left(\sqrt{x}-1\right)^{10}}{x^2\left(x+1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{43}$$

14. **For $\mathbf{V_9(a,b) \leq 2\,V_{12}(a,b)}$:** Let us consider a function $g_{V_9\_V_{12}}(x) = f''_{V_9}(x)/f''_{V_{12}}(x)$. After simplifications, we have

$$g_{V_9\_V_{12}}(x) = \frac{2\left(\sqrt{x}+1\right)^2\left(\begin{array}{c} 35x^2 + 32x^{3/2} + 6\sqrt{x} + 9x + 60x^{5/2} + \\ +6x^{9/2} + 32x^{7/2} + 35x^3 + 9x^4 \end{array}\right)}{\left(x+1\right)^3\left(12x^{5/2} + 27x^2 + 34x^{3/2} + 27x + 12\sqrt{x}\right)},$$

$$\beta_{V_9\_V_{12}} = g_{V_9\_V_{12}}(1) = 2$$

and

$$2V_{12}(a,b) - V_9(a,b) = U_{11}(a,b).$$

Combining the parts 1-11, we get the proof of the inequalities (31). The parts 12-14 give the proof of (32). □

2.4 Third Stage

The proof of above 14 parts give us some new measures. These are given by

$$U_t(P||Q) := \sum_{i=1}^{n} q_i f_{U_t}\left(\frac{p_i}{q_i}\right), \quad t = 1, 2, ..., 11, \tag{44}$$

where $f_{U_t}(x)$, $t = 1, 2, ..., 11$ are as given by (33)-(43) respectively. In all the cases, we have $f_{U_t}(1) = 0$, $t = 1, 2, ..., 11$. By the application of Lemma 1.1, we can say that the above 11 measures are convex. Here below are the second derivatives of the functions (33)-(43), applied frequently in next theorem.

$$f''_{U_1}(x)=\frac{2\left(\sqrt{x}-1\right)^6\left(\sqrt{x}+1\right)^2\left(x^2+x^{3/2}+3x+\sqrt{x}+1\right)}{x^3(x+1)^3},$$

$$f''_{U_2}(x)=\frac{\left(\sqrt{x}-1\right)^6\left(15x+26\sqrt{x}+15\right)}{4x^3\left(x+1\right)^3},$$

$$f''_{U_3}(x)=\frac{\left(\sqrt{x}-1\right)^6\left(\begin{array}{l}15x^4+54x^{7/2}+144x^3+246x^{5/2}+\\+314x^2+246x^{3/2}+144x+54\sqrt{x}+15\end{array}\right)}{2x^{7/2}\left(x+1\right)^3},$$

$$f''_{U_4}(x)=\frac{2\left(\sqrt{x}-1\right)^6\left(\begin{array}{l}297x+960x^2+27+612x^{3/2}+\\+102\sqrt{x}+1156x^{5/2}+612x^{7/2}+\\+102x^{9/2}+27x^5+297x^4+960x^3\end{array}\right)}{x^4\left(x+1\right)^3},$$

$$f''_{U_5}(x)=\frac{2\left(\sqrt{x}-1\right)^6\left(\begin{array}{l}73x+312x^2+3+180x^{3/2}+\\+18\sqrt{x}+312x^3+180x^{7/2}+\\+396x^{5/2}+18x^{9/2}+3x^5+73x^4\end{array}\right)}{x^4\left(x+1\right)^3},$$

$$f''_{U_6}(x)=\frac{\left(\sqrt{x}-1\right)^6\left(\begin{array}{l}462x^{3/2}+462x^{5/2}+90\sqrt{x}+602x^2+\\+252x+90x^{7/2}+15x^4+252x^3+15\end{array}\right)}{4x^{7/2}\left(x+1\right)^3},$$

$$f''_{U_7}(x)=\frac{\left(\sqrt{x}-1\right)^6\left(24x^2+9x^{3/2}-10x+9\sqrt{x}+24\right)}{4x^4},$$

$$f''_{U_8}(x)=\frac{2\left(\sqrt{x}-1\right)^6\left(3x^2+18x^{3/2}+28x+18\sqrt{x}+3\right)}{x^4},$$

$$f''_{U_9}(x)=\frac{\left(\sqrt{x}-1\right)^6\left(12x^2+27x^{3/2}+34x+27\sqrt{x}+12\right)}{2x^4},$$

$$f''_{U_{10}}(x)=\frac{\left(\sqrt{x}-1\right)^8\left(\begin{array}{l}15x^3+40x^{5/2}+77x^2+\\+96x^{3/2}+77x+40\sqrt{x}+15\end{array}\right)}{4x^{7/2}\left(x+1\right)^3}$$

and

$$f''_{U_{11}}(x)=\frac{2\left(\sqrt{x}-1\right)^8\left(\begin{array}{l}3x^4+9x^{7/2}+22x^3+35x^{5/2}+\\+42x^2+35x^{3/2}+22x+9\sqrt{x}+3\end{array}\right)}{x^4\left(x+1\right)^3}.$$

The theorem below connects only the first nine measures. The other two are given later.

Theorem 2.3. *The following inequalities hold:*

$$U_1\le\tfrac{1}{10}U_6\le\tfrac{1}{11}U_3\le\left\{\begin{array}{c}\tfrac{1}{2}U_2\\ \tfrac{1}{56}U_5\end{array}\right\}\le\tfrac{1}{20}U_8\le\tfrac{1}{8}U_9\le\tfrac{1}{2}U_7. \tag{45}$$

Proof. We shall prove the inequalities (45) by parts and shall use the same approach applied in the above theorems. Without specifying, we shall frequently use the second derivatives $f''_{U_t}(x), t = 1, 2, ..., 11$.

1. **For $\mathbf{U_1(a, b \leq \frac{1}{10} U_6(a, b)}$:** Let us consider a function $g_{U_1\_U_6}(x) = f''_{U_1}(x) / f''_{U_6}(x)$. After simplifications, we have

$$g_{U_1\_U_6}(x) = \frac{x\left(3x + 3x^3 + 8x^2 + x^{7/2} + 6x^{3/2} + 6x^{5/2} + \sqrt{x}\right)}{\begin{pmatrix} 462x^{3/2} + 462x^{5/2} + 90\sqrt{x} + 602x^2 \\ +252x + 90x^{7/2} + 15x^4 + 252x^3 + 15 \end{pmatrix}},$$

$$\beta_{U_1\_U_6} = g_{U_1\_U_6}(1) = \frac{1}{10}$$

and

$$\frac{1}{10} U_6(a, b) - U_1(a, b) = \frac{1}{10} U_{10}(a, b).$$

2. **For $\mathbf{U_6(a, b) \leq \frac{10}{11} U_3(a, b)}$:** Let us consider a function $g_{U_6\_U_3}(x) = f''_{U_6}(x) / f''_{U_3}(x)$. After simplifications, we have

$$g_{U_6\_U_3}(x) = \frac{\begin{pmatrix} 462x^{3/2} + 462x^{5/2} + 90\sqrt{x} + 602x^2 + \\ +252x + 90x^{7/2} + 15x^4 + 252x^3 + 15 \end{pmatrix}}{2\begin{pmatrix} 246x^{3/2} + 246x^{5/2} + 54\sqrt{x} + 314x^2 + \\ +144x + 54x^{7/2} + 144x^3 + 15x^4 + 15 \end{pmatrix}},$$

$$\beta_{U_6\_U_3} = g_{U_6\_U_3}(1) = \frac{10}{11}$$

and

$$\frac{10}{11} U_3(a, b) - U_6(a, b) = \frac{9}{11} U_{10}(a, b).$$

3. **For $\mathbf{U_3(a, b) \leq \frac{11}{56} U_5(a, b)}$:** Let us consider a function $g_{U_3\_U_5}(x) = f''_{U_3}(x) / f''_{U_5}(x)$. After simplifications, we have

$$g_{U_3\_U_5}(x) = \frac{x\begin{pmatrix} 246x^{3/2} + 246x^{5/2} + 54\sqrt{x} + 314x^2 + \\ +144x + 54x^{7/2} + 144x^3 + 15x^4 + 15 \end{pmatrix}}{4\begin{pmatrix} 73x^{3/2} + 312x^{5/2} + 3\sqrt{x} + 180x^2 + 73x^{9/2} + \\ +312x^{7/2} + 180x^4 + 396x^3 + 3x^{11/2} + 18x^5 + 18x \end{pmatrix}},$$

$$\beta_{U_3\_U_5} = g_{U_3\_U_5}(1) = \frac{11}{56}$$

and

$$\frac{11}{56} U_5(a, b) - U_3(a, b) = \frac{1}{56} U_{12}(a, b) = \frac{1}{56} b f_{U_{12}}\left(\frac{a}{b}\right),$$

where

$$f_{U_{12}}(x) = \frac{\left(\sqrt{x} - 1\right)^{10}\left(11x - 2\sqrt{x} + 11\right)}{x^2\left(x + 1\right)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{46}$$

4. **For $\mathbf{U_3(a,b) \leq \frac{11}{2} U_2(a,b)}$:** Let us consider a function $g_{U_3\_U_2}(x) = f''_{U_3}(x)/f''_{U_2}(x)$. After simplifications, we have

$$g_{U_3\_U_2}(x) = \frac{2\begin{pmatrix} 246x^{3/2} + 246x^{5/2} + 54\sqrt{x} + 314x^2 + \\ +144x + 54x^{7/2} + 144x^3 + 15x^4 + 15 \end{pmatrix}}{(15x + 26\sqrt{x} + 15)(x+1)^3},$$

$$\beta_{U_3\_U_2} = g_{U_3\_U_2}(1) = \frac{11}{2}$$

and

$$\frac{11}{2} U_2(a,b) - U_3(a,b) = \frac{7}{2} U_{10}(a,b).$$

5. **For $\mathbf{U_2(a,b) \leq \frac{1}{10} U_8(a,b)}$:** Let us consider a function $g_{U_2\_U_8}(x) = f''_{U_2}(x)/f''_{U_8}(x)$. After simplifications, we have

$$g_{U_2\_U_8}(x) = \frac{\sqrt{x}\,(15x + 26\sqrt{x} + 15)}{8\left(3x^2 + 28x + 18x^{3/2} + 18\sqrt{x} + 3\right)},$$

This gives $\beta_{U_2\_U_8} = g_{U_2\_U_8}(1) = \frac{1}{10}$. Let us consider now,

and

$$\frac{1}{10} U_8(a,b) - U_2(a,b) = \frac{1}{10} U_{13}(a,b) = \frac{1}{10} b\, f_{U_{13}}\left(\frac{a}{b}\right),$$

where

$$f_{U_{13}}(x) = \frac{(\sqrt{x} - 1)^{10}}{x^2} > 0, \quad \forall x > 0,\ x \neq 1. \tag{47}$$

6. **For $\mathbf{U_5(a,b) \leq \frac{14}{5} U_8(a,b)}$:** Let us consider a function $g_{U_5\_U_8}(x) = f''_{U_5}(x)/f''_{U_8}(x)$. After simplifications, we have

$$g_{U_5\_U_8}(x) = \frac{\begin{pmatrix} 73x^{3/2} + 312x^{5/2} + 3\sqrt{x} + 180x^2 + 18x + 73x^{9/2} \\ +312x^{7/2} + 180x^4 + 396x^3 + 3x^{11/2} + 18x^5 \end{pmatrix}}{\left(3x^{5/2} + 28x^{3/2} + 18x^2 + 18x + 3\sqrt{x}\right)(x+1)^3},$$

$$\beta_{U_5\_U_8} = g_{U_5\_U_8}(1) = \frac{14}{5}$$

and

$$\frac{14}{5} U_8(a,b) - U_5(a,b) = \frac{9}{5} U_{14}(a,b) = \frac{9}{5} b\, f_{U_{14}}\left(\frac{a}{b}\right),$$

where

$$f_{U_{14}}(x) = \frac{(\sqrt{x} - 1)^{10}(x + 10\sqrt{x} + 1)}{x^2(x+1)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{48}$$

7. **For $\mathbf{U_8(a,b) \leq \frac{5}{2} U_9(a,b)}$:** Let us consider a function $g_{U_8\_U_9}(x) = f''_{U_8}(x)/f''_{U_9}(x)$. After simplifications, we have

$$g_{U_8\_U_9}(x) = \frac{4\left(3x^2 + 18x^{3/2} + 28x + 18\sqrt{x} + 3\right)}{12x^2 + 27x^{3/2} + 34x + 27\sqrt{x} + 12},$$

$$\beta_{U_8\_U_9} = g_{U_8\_U_9}(1) = \frac{5}{2}$$

and

$$\frac{5}{2}U_9(a,b) - U_8(a,b) = \frac{3}{2}U_{13}(a,b),$$

8. **For $\mathbf{U_9(a,b) \leq 4\,U_7(a,b)}$:** Let us consider a function $g_{U_9\_U_7}(x) = f''_{U_9}(x)/f''_{U_7}(x)$. After simplifications, we have

$$g_{U_9\_U_7}(x) = \frac{2\left(12x^2 + 27x^{3/2} + 34x + 27\sqrt{x} + 12\right)}{24x^2 + 9x^{3/2} - 10x + 9\sqrt{x} + 24},$$

$$\beta_{U_9\_U_7} = g_{U_9\_U_7}(1) = 4$$

and

$$4U_7(a,b) - U_9(a,b) = 3U_{13}(a,b).$$

Combining the parts 1-8, we get the proof of the inequalities (45). $\square$

2.5 Forth Stage

Still, we have more measures to compares, i.e., U_{10} to U_{14}. This comparison is given in the theorem below. Here below are the second derivatives of the functions given by (46)-(48).

$$f''_{U_{12}}(x) = \frac{(\sqrt{x}-1)^8 \begin{pmatrix} 15x^3 + 40x^{5/2} + 77x^2 + \\ +96x^{3/2} + 77x + 40\sqrt{x} + 15 \end{pmatrix}}{4x^{7/2}\,(x+1)^3},$$

$$f''_{U_{13}}(x) = \frac{(\sqrt{x}-1)^8 \begin{pmatrix} 15x^3 + 40x^{5/2} + 77x^2 + \\ +96x^{3/2} + 77x + 40\sqrt{x} + 15 \end{pmatrix}}{4x^{7/2}\,(x+1)^3},$$

and

$$f''_{U_{14}}(x) = \frac{2\,(\sqrt{x}-1)^8 \begin{pmatrix} 3x^4 + 9x^{7/2} + 22x^3 + 35x^{5/2} + \\ +42x^2 + 35x^{3/2} + 22x + 9\sqrt{x} + 3 \end{pmatrix}}{x^4\,(x+1)^3}.$$

Theorem 2.4. *The following inequalities hold:*

$$U_{10} \leq \tfrac{1}{12}U_{14} \leq \tfrac{1}{4}U_{11} \leq \tfrac{1}{2}U_{13} \leq \tfrac{1}{20}U_{12}. \tag{49}$$

Proof. We shall prove the above theorem by parts.

1. **For $\mathbf{U_{10}(a,b) \leq \frac{1}{12}\,U_{14}(a,b)}$:** Let us consider a function $g_{U_{10}\_U_{14}}(x) = f''_{U_{10}}(x)/f''_{U_{14}}(x)$. After simplifications, we have

$$g_{U_{10}\_U_{14}}(x) = \frac{x \begin{pmatrix} 15 + 77x^2 + 77x + 40x^{5/2} + \\ +15x^3 + 40\sqrt{x} + 96x^{3/2} \end{pmatrix}}{8 \begin{pmatrix} 62x^{3/2} + 138x^{5/2} + 3\sqrt{x} + 112x^2 + \\ 24x + 3x^{9/2} + 62x^{7/2} + 24x^4 + 112x^3 \end{pmatrix}},$$

$$\beta_{U_{10}\_U_{14}} = g_{U_{10}\_U_{14}}(1) = \frac{1}{12}$$

and

$$\frac{1}{12}U_{14}(a,b) - U_{10}(a,b) = \frac{1}{12}U_{15}(a,b) = \frac{1}{12}b\,f_{U_{15}}\left(\frac{a}{b}\right).$$

where

$$f_{U_{15}}(x) = \frac{\left(\sqrt{x}-1\right)^{12}}{x^2\,(x+1)} > 0, \quad \forall x > 0,\ x \neq 1. \tag{50}$$

2. **For $\mathbf{U_{14}(a,b) \leq 3\,U_{11}(a,b)}$:** Let us consider a function $g_{U_{14}\_U_{11}}(x) = f''_{U_{14}}(x)/f''_{U_{11}}(x)$. After simplifications, we have

$$g_{U_{14}\_U_{11}}(x) = \frac{\left(\begin{array}{l} 62x^{3/2} + 138x^{5/2} + 3\sqrt{x} + 112x^2 + \\ +24x + 3x^{9/2} + 62x^{7/2} + 24x^4 + 112x^3 \end{array}\right)}{\left(\begin{array}{l} 22x^{3/2} + 42x^{5/2} + 3\sqrt{x} + 35x^2 + \\ +9x + 3x^{9/2} + 22x^{7/2} + 9x^4 + 35x^3 \end{array}\right)},$$

$$\beta_{U_{14}\_U_{11}} = g_{U_{14}\_U_{11}}(1) = 3$$

and

$$3U_{11}(a,b) - U_{14}(a,b) = 2U_{15}(a,b).$$

3. **For $\mathbf{U_{11}(a,b) \leq 2\,U_{13}(a,b)}$:** Let us consider a function $g_{U_{11}\_U_{13}}(x) = f''_{U_{11}}(x)/f''_{U_{13}}(x)$. After simplifications, we have

$$g_{U_{11}\_U_{13}}(x) = \frac{4\left(\begin{array}{l} 22x + 42x^2 + 3 + 35x^{3/2} + \\ +9\sqrt{x} + 3x^4 + 22x^3 + 9x^{7/2} + 35x^{5/2} \end{array}\right)}{3\left(4x + 7\sqrt{x} + 4\right)(x+1)^3},$$

$$\beta_{U_{11}\_U_{13}} = g_{U_{11}\_U_{13}}(1) = 2$$

and

$$2U_{13}(a,b) - U_{11}(a,b) = U_{15}(a,b).$$

4. **For $\mathbf{U_{13}(a,b) \leq \frac{1}{10}U_{12}(a,b)}$:** Let us consider a function $g_{U_{13}\_U_{12}}(x) = f''_{U_{13}}(x)/f''_{U_{12}}(x)$. After simplifications, we have

$$g_{U_{13}\_U_{12}}(x) = \frac{3\left(4x + 7\sqrt{x} + 4\right)(x+1)^3}{4\left(\begin{array}{l} 122x + 174x^2 + 33 + 154x^{3/2} + 54\sqrt{x} + \\ +33x^4 + 122x^3 + 54x^{7/2} + 154x^{5/2} \end{array}\right)},$$

$$\beta_{U_{13}\_U_{12}} = g_{U_{13}\_U_{12}}(1) = \frac{1}{10}$$

and

$$\frac{1}{10}U_{12}(a,b) - U_{13}(a,b) = \frac{1}{10}U_{15}(a,b).$$

□

Remark 2.3. *Interestingly, in all the four cases we are left with only a single measure, i.e., $U_{15}(a,b)$ given by*

$$U_{15}(a,b) = \frac{\left(\sqrt{a}-\sqrt{b}\right)^{12}}{(ab)^2\,(a+b)}, \quad a,b > 0, \quad a \neq b. \tag{51}$$

2.6 Equivalent Expressions

The measures appearing in the proof of the Theorems 2.2-2.4 can be written in terms of the measures appearing in the inequalities (8). Here below are equivalent versions of these measures.

• **Measures appearing in Theorem 2.2**. We can write

$$\begin{aligned}
V_1 &= K + 26\Delta - 48D_{CN} \\
&= K + 30D_{CN} - 26D_{CG} \\
&= K + 14D_{CG} - 30D_{RG} \\
&= K + 12D_{RG} - 28hD_{RG}, \\
V_2 &= \Psi + 64h - 4\Delta - 8K, \\
V_3 &= \Psi + 108\Delta - 192D_{CN} \\
&= \Psi + 132D_{CN} - 108D_{CG} \\
&= \Psi + 68D_{CG} - 132D_{RG} \\
&= \Psi + 72D_{RG} - 136h, \\
V_4 &= \Psi + 32h - 6K, \\
V_5 &= 2\left(F + 6K - 4\Delta - 3\Psi\right), \\
V_6 &= 2\left(F + 164\Delta - 288D_{CN}\right) \\
&= 2\left(F + 204D_{CN} - 164D_{CG}\right) \\
&= 2\left(F + 108D_{CG} - 204D_{RG}\right) \\
&= 2\left(F + 120D_{RG} - 216h\right), \\
V_7 &= 2\left(F - 10K + 64h\right), \\
V_8 &= 2\left(F + 2K - 2\Psi\right), \\
V_9 &= L + 16K - 16\Delta - 8F, \\
V_{10} &= L + 880\Delta - 1536D_{CN} \\
&= L + 1104D_{CN} - 880D_{CG} \\
&= L + 592D_{CG} - 1104D_{RG} \\
&= L + 6724D_{RG} - 1184h \\
V_{11} &= L + 384h - 56K, \\
V_{12} &= L + 12\Psi - 8K - 12F, \\
V_{13} &= L + 16K - 12\Psi, \\
V_{14} &= L + 4\Psi - 8F.
\end{aligned}$$

• **Measures appearing in Theorem 2.3 and 2.4**. We can write

$$\begin{aligned}
U_1 &= \Psi + 192D_{CN} - 100\Delta - 8K, \\
U_2 &= 2\left(F + 14K - 64h - 4\Psi\right), \\
U_3 &= 4F + 576D_{CN} - 316\Delta - 9\Psi, \\
U_4 &= 9L + 4608D_{CN} - 2576\Delta - 64F, \\
U_5 &= \tfrac{1}{7}\left(7L + 6144h + 13824D_{CN} - 896K - 7920\Delta\right), \\
U_6 &= \tfrac{2}{5}\left(5F + 576h + 1152D_{CN} - 90K - 656\Delta\right), \\
U_7 &= L + 384h + 36\Psi - 92K - 18F, \\
U_8 &= L + 160K - 36\Psi - 768h, \\
U_9 &= L + 12\Psi - 8K - 12F, \\
U_{10} &= 2\left(F + 22K + 4\Delta - 5\Psi - 128h\right), \\
U_{11} &= L + 16\Delta + 24\Psi - 16F - 32K, \\
U_{12} &= \tfrac{1}{7}\left(77L - 9856K + 67584h - 73728D_{CN} + 36752\Delta - 1568F + 3528\Psi\right), \\
U_{13} &= L + 44\Psi - 120K + 512h - 20F \\
U_{14} &= \tfrac{1}{7}\left(7L - 392\Psi + 2240K - 11776h - 7680D_{CN} + 4400\Delta\right), \\
U_{15} &= \tfrac{1}{7}\left(7L + 448\Delta - 1456K + 9728h - 7680D_{CN} + 3728\Delta - 168F\right).
\end{aligned}$$

3 Generating Divergence Measures

Some of the measures given in Section 2 can be written in generating forms. Let us see below these generating measures.

3.1 First Generalization of Triangular Discrimination

For all $(a, b) \in \mathrm{R}^2_+$, let consider the following measures

$$\Delta^1_t(a,b) = \frac{(a-b)^2\left(\sqrt{a}-\sqrt{b}\right)^{2t}}{(a+b)\left(\sqrt{ab}\right)^t}, \quad t = 0, 1, 2, 3, ... \tag{52}$$

In particular, we have

$$\begin{aligned}
\Delta^1_0 &= \Delta = \frac{(a-b)^2}{(a+b)}, \\
\Delta^1_1 &= D^{15}_{W_6W_1} = K - 2\Delta = \frac{(a-b)^2\left(\sqrt{a}-\sqrt{b}\right)^2}{(a+b)\sqrt{ab}}, \\
\Delta^1_2 &= \Psi - 4K + 4\Delta = \frac{(a-b)^2\left(\sqrt{a}-\sqrt{b}\right)^4}{ab\,(a+b)}
\end{aligned}$$

and

$$\Delta_3^1 = V_5 = 2\left(F + 6K - 4\Delta - 3\Psi\right) = \frac{(a-b)^2\left(\sqrt{a}-\sqrt{b}\right)^6}{(ab)^{3/2}\,(a+b)}.$$

The expression (52) gives first generalization of the measure $\Delta(a,b)$. Let us prove now its convexity. We can write $\Delta_t^1(a,b) = bf_{\Delta_t^1}(a/b)$, $t \in \mathrm{N}$, where

$$f_{\Delta_t^1}(x) = \frac{(x-1)^2\left(\sqrt{x}-1\right)^{2t}}{(x+1)\left(\sqrt{x}\right)^t}. \tag{53}$$

The second order derivative of the function $f_{\Delta_t^1}(x)$ is given by

$$f''_{\Delta_t^1}(x) = \frac{\left(\sqrt{x}-1\right)^{2t}}{4x^2\,(x+1)^3\left(\sqrt{x}\right)^t} \times A_1(x,t),$$

where

$$A_1(x,t) = \left(\begin{array}{l} t\,(t+2)\left(x^4+1\right) + 2t\,(2t+1)\sqrt{x}\left(x^3+1\right) + \\ +4t\,(2t+3)\,x\left(x^2+1\right) + 4\left(7t^2+10t+16\right)x^2 \\ +2t\,(6t+11)\,x^{3/2}\,(x+1) \end{array}\right).$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{\Delta_t^1}(x) > 0$. Also we have $f_{\Delta_t^1}(1) = 0$. In view of Lemma 1.1, the measure $\Delta_t^1(a,b)$ is convex for all $(a,b) \in \mathrm{R}_+^2$, $t \in \mathrm{N}$.

Now, we shall present exponential representation of the measure (52) based on the function given by (53). Let us consider a linear combination of convex functions,

$$f_{\Delta^1}(x) = a_0 f_{\Delta_0^1}(x) + a_1 f_{\Delta_1^1}(x) + a_2 f_{\Delta_2^1}(x) + a_3 f_{\Delta_3^1}(x) + ...$$

i.e.,

$$f_{\Delta^1}(x) = a_0\frac{(x-1)^2}{x+1} + a_1\frac{(x-1)^2\left(\sqrt{x}-1\right)^2}{(x+1)\sqrt{x}} + a_2\frac{(x-1)^2\left(\sqrt{x}-1\right)^4}{x\,(x+1)} + \\ +a_3\frac{(x-1)^2\left(\sqrt{x}-1\right)^6}{(x)^{3/2}\,(x+1)} + ...,$$

where $a_0, a_1,\ a_2,\ a_3, ...$are the constants. For simplicity let us choose,

$$a_0 = \frac{1}{0!},\ a_1 = \frac{1}{1!},\ a_2 = \frac{1}{2!},\ a_3 = \frac{1}{3!}, ...$$

Thus we have

$$\begin{aligned} f_{\Delta^1}(x) &= \frac{1}{0!}\frac{(x-1)^2}{x+1} + \frac{1}{1!}\frac{(x-1)^2\left(\sqrt{x}-1\right)^2}{(x+1)\sqrt{x}} + \frac{1}{2!}\frac{(x-1)^2\left(\sqrt{x}-1\right)^4}{x\,(x+1)} + \\ &\quad + \frac{1}{3!}\frac{(x-1)^2\left(\sqrt{x}-1\right)^6}{(x)^{3/2}\,(x+1)} + ... \\ &= \frac{(x-1)^2}{x+1}\left[\frac{1}{0!} + \frac{1}{1!}\left(\frac{(\sqrt{x}-1)^2}{\sqrt{x}}\right)^1 + \frac{1}{2!}\left(\frac{(\sqrt{x}-1)^2}{\sqrt{x}}\right)^2 + \frac{1}{3!}\left(\frac{(\sqrt{x}-1)^2}{\sqrt{x}}\right)^3 + ...\right]. \end{aligned}$$

This gives us

$$f_{\Delta^1}(x) = \frac{(x-1)^2}{x+1} \exp\left(\frac{(x-1)^2}{\sqrt{x}}\right). \tag{54}$$

As a consequence of (54), we have the following exponential representation of triangular discrimination

$$E_{\Delta^1}(a,b) = b\, f_{\Delta^1}(a/b) = \frac{(a-b)^2}{a+b} \exp\left(\frac{(a-b)^2}{\sqrt{ab}}\right), \quad (a,b) \in \mathrm{R}_+^2. \tag{55}$$

3.2 Second Generalization of Triangular Discrimination

For all $(a,b) \in \mathrm{R}_+^2$, let consider the following measures

$$\Delta_t^2(a,b) = \frac{(a-b)^{2(t+1)}}{(a+b)\,(ab)^t}, \quad t = 0,1,2,3,... \tag{56}$$

In particular, we have

$$\Delta_0^2 = \Delta = \frac{(a-b)^2}{a+b}$$

and

$$\Delta_1^2 = 2D_{W_7W_1}^{21} = \Psi - 4\Delta = \frac{(a-b)^4}{ab\,(a+b)}.$$

The expression (56) gives us a second generalization of the measure $\Delta(a,b)$. Let us prove now its convexity. We can write $\Delta_t^2(a,b) = b\, f_{\Delta_t^2}(a/b)$, $t \in \mathrm{N}$, where

$$f_{\Delta_t^2}(x) = \frac{(\sqrt{x}-1)^{2(t+1)}}{(x+1)\,x^t}.$$

The second order derivative of the function $f_{\Delta_t^2}(x)$ is given by

$$f''_{\Delta_t^2}(x) = \frac{(x-1)^{2t}}{(x+1)^3\,x^{t+2}} \times A_2(x,t),$$

where

$$A_2(x,t) = t\,(t+1)\left(x^4+1\right) + 2t\,(2t+3)\,x\left(x^2+1\right) + 2\left(3t^2+5t+4\right)x^2.$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{\Delta_t^2}(x) > 0$. Also we have $f_{\Delta_t^2}(1) = 0$. In view of Lemma 1.1, the measure $\Delta_t^2(a,b)$ is convex for all $(a,b) \in \mathrm{R}_+^2$, $t \in \mathrm{N}$.

Following similar lines of (54) and (55), the exponential representation of the measure $\Delta_t^2(a,b)$ is given by

$$E_{\Delta^2}(a,b) = \frac{(a-b)^2}{a+b} \exp\left(\frac{(a-b)^2}{ab}\right), \quad (a,b) \in \mathrm{R}_+^2.$$

3.3 First Generalization of the Measure $\mathbf{K}(\mathrm{a}, \mathrm{b})$

For all $(a, b) \in \mathrm{R}_+^2$, let consider the following measures

$$K_t^1(a,b) = \frac{(a-b)^2 \left(\sqrt{a}-\sqrt{b}\right)^{2t}}{\left(\sqrt{ab}\right)^{t+1}}, \quad t = 0, 1, 2, 3, ... \tag{57}$$

In particular, we have

$$K_0^1 = K = \frac{(a-b)^2}{\sqrt{ab}},$$

$$K_1^1 = 2D_{W_7W_6}^{16} = \Psi - 2K = \frac{(a-b)^2 \left(\sqrt{a}-\sqrt{b}\right)^2}{ab},$$

$$K_2^1 = V_8 = 2\left(F + 2K - 2\Psi\right) = \frac{(a-b)^2 \left(\sqrt{a}-\sqrt{b}\right)^4}{(ab)^{3/2}}$$

and

$$K_3^1 = V_{12} = L + 12\Psi - 8K - 12F = \frac{(a-b)^2 \left(\sqrt{a}-\sqrt{b}\right)^6}{(ab)^2}. \tag{58}$$

The expression (59) gives first parametric generalization the measure $K(a, b)$ given by (3). Let us prove now its convexity. We can write $K_t^1(a, b) = b\, f_{K_t^1}(a/b)$, $t \in \mathrm{N}$, where

$$f_{K_t^1}(x) = \frac{(x-1)^2 \left(\sqrt{x}-1\right)^{2t}}{\left(\sqrt{x}\right)^{t+1}}.$$

The second order derivative of the function $f_{K_t^1}(x)$ is given by

$$f''_{K_t^1}(x) = \frac{\left(\sqrt{x}-1\right)^{2t}}{4x^2 \left(\sqrt{x}\right)^{t+1}} \times A_3(x,t),$$

where

$$A_3(x,t) = (t+1)(t+3)\left(x^2+1\right) + 2t\,(2t+3)\sqrt{x}\,(x+1) + 2\left(3t^2+2t+1\right)x.$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{K_t^1}(x) > 0$. Also we have $f_{K_t^1}(1) = 0$. In view of Lemma 1.1, the measure $K_t^1(a, b)$ is convex for all $(a, b) \in \mathrm{R}_+^2$, $t \in \mathrm{N}$.

Following similar lines of (54) and (55), the exponential representation of the measure $K_t^1(a, b)$ is given by

$$E_{K^1}(a,b) = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\sqrt{ab}} \exp\left(\frac{(a-b)^2}{\sqrt{ab}}\right), \quad (a,b) \in \mathrm{R}_+^2.$$

3.4 Second Generalization of the Measure $\mathbf{K}(\mathrm{a}, \mathrm{b})$

For all $(a,b) \in \mathrm{R}_+^2$, let consider the following measures

$$K_t^2(a,b) = \frac{(a-b)^{2(t+1)}}{\left(\sqrt{ab}\right)^{2t+1}},\ t = 0,1,2,3,... \tag{59}$$

In particular, we have

$$K_0^2 = K = \frac{(a-b)^2}{\sqrt{ab}}$$

and

$$K_1^2 = 4D_{W_8 W_6}^{23} = F - 2K = \frac{(a-b)^4}{(ab)^{3/2}}.$$

The expression (59) gives second generalization the measure $K(a,b)$ given by (3). Let us prove now its convexity. We can write $K_t^2(a,b) = b\, f_{K_t^2}(a/b)$, $t \in \mathrm{N}$, where

$$f_{K_t^2}(x) = \frac{(x-1)^{2(t+1)}}{\left(\sqrt{x}\right)^{2t+1}}.$$

The second order derivative of the function $f_{K_t^2}(x)$ is given by

$$f''_{K_t^2}(x) = \frac{(x-1)^{2t}}{4x^2\left(\sqrt{x}\right)^{2t+1}} \times A_4(x,t),$$

where

$$A_4(x,t) = (2t+1)\left[2tx^2 + 3x^2 + 2\,(2t+1)\,x + 2t + 3\right].$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{K_t^2}(x) > 0$. Also we have $f_{K_t^2}(1) = 0$. In view of Lemma 1.1, the measure $K_t^2(a,b)$ is convex for all $(a,b) \in \mathrm{R}_+^2$, $t \in \mathrm{N}$.

Following similar lines of (54) and (55), the exponential representation of the measure $K_t^2(a,b)$ is given by

$$E_{K^2}(a,b) = \frac{(a-b)^2}{\sqrt{ab}} \exp\left(\frac{(a-b)^2}{ab}\right), \quad (a,b) \in \mathrm{R}_+^2.$$

3.5 Generalization of Hellingar's Discrimination

For all $(a,b) \in \mathrm{R}_+^2$, let consider the following measures

$$h_t(a,b) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^{2(t+1)}}{\left(\sqrt{ab}\right)^t},\ t \in \mathrm{N} \tag{60}$$

In particular, we have

$$h_0 = 2h = \left(\sqrt{a} - \sqrt{b}\right)^2,$$

$$h_1 = D^{11}_{W_6 W_5} = K - 8h = \frac{\left(\sqrt{a} - \sqrt{b}\right)^4}{\sqrt{ab}},$$

$$h_2 = V_4 = \Psi + 32h - 6K = \frac{\left(\sqrt{a} - \sqrt{b}\right)^6}{ab},$$

$$h_3 = U_2 = 2\left(F + 14K - 4\Psi - 64h\right) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^8}{(ab)^{3/2}}$$

and

$$h_4 = U_{13} = L + 44\Psi - 120K + 512h - 20F = \frac{\left(\sqrt{a} - \sqrt{b}\right)^{10}}{(ab)^2}.$$

The measure (60) give generalized Hellingar's discrimination. Let us prove now its convexity. We can write $h_t(a,b) = b\, f_{h_t}(a/b)$, $t \in \mathrm{N}$, where

$$f_{h_t}(x) = \frac{\left(\sqrt{x} - 1\right)^{2(t+1)}}{\left(\sqrt{x}\right)^t}.$$

The second order derivative of the function $f_{h_t}(x)$ is given by

$$f''_{h_t}(x) = \frac{\left(\sqrt{x} - 1\right)^{2t}}{4\left(\sqrt{x}\right)^{t+5}} \times A_5(x,t),$$

where

$$A_5(x,t) = t\,(t+2)\sqrt{x}\,(x+1) + 2\left(t^2 + t + 1\right)x.$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{h_t}(x) > 0$. Also $f_{h_t}(1) = 0$. In view of Lemma 1.1, the measure $h_t(a,b)$ is convex for all $(a,b) \in \mathrm{R}^2_+$, $t \in \mathrm{N}$.

Following similar lines of (54) and (55), the exponential representation of the measure $h_t(a,b)$ is given by

$$E_h(a,b) = \left(\sqrt{a} - \sqrt{b}\right)^2 \exp\left(\frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{\sqrt{ab}}\right), \quad (a,b) \in \mathrm{R}^2_+.$$

3.6 New Measure

For all $P,\ Q \in \Gamma_n$, let consider the following measures

$$M_t(a,b) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^{2(t+2)}}{(a+b)\left(\sqrt{ab}\right)^t}, \quad t = 0, 1, 2, 3, \ldots \tag{61}$$

In particular, we have

$$M_0 = \frac{7}{2} D^1_{W_2 W_1} = 12 D_{CN} - 7\Delta = \frac{\left(\sqrt{a} - \sqrt{b}\right)^4}{(a+b)},$$

$$M_1 = V_1 = K + 26\Delta - 48 D_{CN} = \frac{\left(\sqrt{a} - \sqrt{b}\right)^6}{(a+b)\sqrt{ab}},$$

$$M_2 = U_1 = \Psi + 192 D_{CN} - 100\Delta - 8K = \frac{\left(\sqrt{a} - \sqrt{b}\right)^8}{ab\,(a+b)},$$

$$M_3 = U_{10} = 2\left(F + 22K + 4\Delta - 5\Psi - 128h\right) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^{10}}{(a+b)\,(ab)^{3/2}}$$

and

$$M_4 = U_{15} = \frac{1}{7}\left(\begin{array}{l} 7L + 448\Delta - 1456K + 3728\Delta + \\ +9728h - 7680 D_{CN} - 168F \end{array}\right) = \frac{\left(\sqrt{a} - \sqrt{b}\right)^{12}}{(a+b)\,(ab)^2}.$$

Let us prove now the convexity of the measure (3.17). We can write $M_t(a,b) = b\, f_{M_t}(a/b)$, $t \in \mathrm{N}$, where

$$f_{M_t}(x) = \frac{\left(\sqrt{x} - 1\right)^{2(t+2)}}{(x+1)\left(\sqrt{x}\right)^t}.$$

The second order derivative of the function $f_{M_t}(x)$ is given by

$$f''_{M_t}(x) = \frac{(x-1)^{2t+2}}{4\,(x+1)^3\left(\sqrt{x}\right)^{t+5}} \times A_6(x,t),$$

where

$$A_6(x,t) = \left(\begin{array}{l} 2\left(t^2 + 3t + 2\right)x\left(x^2+1\right) + 4\left(t^2 + 3t + 6\right)x^2 + \\ +t\,(t+2)\sqrt{x}\left(x^3+1\right) + \left(3t^2 + 14t + 8\right)x^{3/2}\,(x+1) \end{array}\right).$$

For all $t \geq 0$, $x > 0$, $x \neq 1$, we have $f''_{M_t}(x) > 0$. Also we have $f_{M_t}(1) = 0$. In view of Lemma 1.1, the measure $M_t(a,b)$ is convex for all $(a,b) \in \mathrm{R}^2_+$, $t \in \mathrm{N}$.

Following similar lines of (54) and (55), the exponential representation of the measure $M_t(a,b)$ is given by

$$E_M(a,b) = \frac{(a-b)^4}{a+b} \exp\left(\frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{\sqrt{ab}}\right), \quad (a,b) \in \mathrm{R}^2_+.$$

Remark 3.1. *(i) The first ten measures appearing in the second pyramid represents the same measure (14) and is the same as M_0. The last measure given by (52) is the same as M_4. The measure (52) is the only that appears in all the four parts of the last Theorem 2.4. Both these measures generates an interesting measure (61).*

(ii) The measure K_1^1 appears in the work of Dragomir et al. [1]. An improvement over this work can be seen in Taneja [7].

(iii) Following similar lines of (54) and (55), the exponential representation of the principal measure $L_t(a,b)$ appearing in (5) is given by

$$E_\Delta(a,b) = \frac{2\,(a-b)^2}{a+b}\exp\left(\frac{a+b}{2\sqrt{ab}}\right),\ (a,b)\in \mathrm{R}_+^2. \tag{62}$$

We observe that the expression (62) is little different from the one obtained above in six parts. Applications of the generating measures (5), (52), (56), (57), (59), (60) and (61) along with their exponential representations shall be dealt elsewhere.

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